SCHEDULING AND ROUTING IN
TRANSPORTATION AND DISTRIBUTION
SYSTEMS: FORMULATIONS
AND NEW RELAXATIONS

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ABSTRACT

New formulations are presented for the traveling salesman problem, and their relationship to previous formulations is investigated. The new formulations are extended to include a variety of transportation scheduling problems, such as the multi-traveling salesman problem, the delivery problem, the school bus problem, and the dial-a-bus problem. A Benders decomposition procedure is applied to the new formulations and the resulting computational procedure is seen to be identical to previous methods for solving the traveling salesman problem. The new formulations are also shown to have a dual relationship with an earlier formulation due to Miller, Tucker and Zemlin. Three Lagrangean relaxations are proposed for the new formulation, one of which is shown to be identical to the l-tree relaxation suggested by Held and Karp. New possibilities are indicated for developing new branch and bound procedures for scheduling and routing in transportation systems.
1. Introduction

The travelling salesman problem is to find a minimum cost tour of a set of cities or nodes in which the tour visits each node exactly once. This problem has many applications in sequencing and scheduling theory, and has been studied extensively in the operations research literature (e.g., [3], [6], [8], [24-28], [31], [34]). One of the first compact mathematical formulations to the travelling salesman problem was given by Miller, et. al. [31] in 1960 as:

**Problem P0:**

Find variables $x_{ij}$ and $u_i$ for $i,j=1,2,...,n$ that minimize

$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$  \hspace{1cm} (1)

subject to:

$$\sum_{i=1}^{n} x_{ij} = 1 \quad j=1,2,...,n, \hspace{1cm} (2)$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad i=1,2,...,n, \hspace{1cm} (3)$$

$$u_i - u_j + n x_{ij} \leq n - 1 \quad i,j=2,...,n \text{ if } j \neq i \hspace{1cm} (4)$$

$$x_{ij} = 0,1 \quad \forall i,j \hspace{1cm} (5)$$

*We assume throughout that $c_{ii} = \infty$, $i=1,2,...,n.$*
This is a mixed integer programming formulation with $n^2$ zero-one variables and $n-1$ continuous variables for $n$ being the number of nodes in the system. The variables $X_{ij}$ are zero-one variables to denote the inclusion of the arc connecting node $i$ to node $j$ in the optimal tour, while $U_i$ denotes the order in which the nodes are visited in the optimal tour. In spite of the compactness of this formulation, we know of no algorithms or computational test results in the open literature which have used this formulation as a basis for solving the traveling salesman problem. One of the major drawbacks of this formulation is that it is limited to the traveling salesman problem only and cannot be easily extended to other transportation scheduling problems which are related to the traveling salesman problem such as the multi-traveling salesman problem, the delivery problem, the school bus problem, the multi-terminal delivery problem, or the static dial-a-bus problem.

In this paper we present a new formulation for the traveling salesman problem. This formulation can be shown to have a dual relationship with the Miller, et. al. formulation PO. One advantage of this new formulation is the fact that it can be extended to model the related transportation scheduling problems mentioned above. In addition, this formulation suggests new relaxations and algorithms for solving routing and scheduling problems.

The remainder of the paper is organized as follows: In the next section, the new formulation of the traveling salesman problem is given. This formulation is extended in Section 3 to a general class of transportation scheduling problems. In Section 4 we establish the dual relationship with PO; we also show that the application of Benders decomposition to the new formulation results in the standard formulation of the traveling
salesman problem given by Dantzig, Fulkerson and Johnson [8]. In Section 5, we present three Lagrangean relaxations suggested by the new formulations and indicate the current research efforts exploring these relaxations.
2. New Formulation for the Traveling Salesman Problem

The new formulation for the traveling salesman problem is as follows:

Problem P1:

Find variables $X_{ij}, y_{ij}$ $i,j=1,2,...,n$ that minimize

$$Z = \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}X_{ij} \right\}$$

subject to:

$$\sum_{i=1}^{n} X_{ij} = 1 \quad \text{for } j=1,2,...,n \tag{7}$$

$$\sum_{j=1}^{n} X_{ij} = 1 \quad \text{for } i=1,2,...,n \tag{8}$$

$$\sum_{j=1}^{n} y_{ji} - \sum_{j=2}^{n} y_{ij} = 1 \quad \text{for } i=2,...,n \tag{9}$$

$$y_{ij} \leq (n-1)X_{ij} \quad \text{for } i=1,...,n \tag{10}$$

$$X_{ij} = 0,1, y_{ij} \geq 0 \quad \text{for } j=2,...,n \text{ if } i \neq j \tag{11}$$

For fixed values of $X_{ij}$, the constraints given in (9) and (10) form a network flow problem, and therefore the optimal $y_{ij}$ values are integer.
**Lemma:** Problem $P1$ solves the traveling salesman problem.

**Proof:** To show that Problem $P1$ solves the traveling salesman problem is sufficient to show that the feasible region for $P1$ given by (7-11) is equivalent to the set of tours for the traveling salesman problem.

Consider any tour $\tau = \{i_0 = 1, i_1, i_2, \ldots, i_{n-1}, i_n = 1\}$. It is easy to see that the corresponding feasible solution in $P1$ has $x_{i_k, i_{k+1}} = 1$ and $y_{i_k, i_{k+1}} = n-k$ for $k=0, 1, \ldots, n-1$, with $x_{ij} = 0$ and $y_{ij} = 0$ otherwise.

Now consider any feasible solution $\{x_{ij}, y_{ij}\}$ to $P1$. Since $\{x_{ij}\}$ are zero-one variables which satisfy the assignment constraints (7-8), their values correspond to an extreme point of the assignment problem. It is well known that the positive variables in an extreme point of the assignment polytope form distinct loops of arcs in a graph that contains arc $(i, j)$ iff $x_{ij} = 1$, and each node appears in exactly one loop. If only one loop is given by the values of $\{x_{ij}\}$, this loop corresponds to a tour for the traveling salesman problem. We will show that the solution $\{x_{ij}, y_{ij}\}$ defines exactly one loop. Assume that more than one loop is given by a solution and consider a loop consisting of the sequence of nodes $\{i_1, i_2, \ldots, i_r, i_1\}$ which does not contain node 1. Let $y_{i_1, i_1} = f, f \geq 0$; from (9-10) it follows that $y_{i_k, i_{k+1}} = f + r - k$ for $k=1, 2, \ldots, r-1$. Therefore we have

$$
\sum_{j=1, j \neq i_1}^{n} y_{ji_1} - \sum_{j=2}^{n} y_{i_1j} = f - (f + r - 1) = 1 - r
$$

which contradicts (9). Thus no loops exist that do not contain node 1; since node 1 is contained in exactly one loop, at most one loop (tour) is generated. This completes the proof.
In this formulation, the flows, $y_{ij}$, along the tour are strictly *decreasing* (assuming that the tour starts and ends in node 1). An alternate formulation would have these flow be strictly *increasing*. This is obtained by replacing the constraints in (9) by:

$$\sum_{j=1}^{n} y_{ij} - \sum_{j=2}^{n} y_{ji} = 1 \quad i=2, \ldots, n.$$  \hspace{1cm} (12)

In [13], we present a second, more compact formulation for the traveling salesman problem which is similar to P1. Essentially, we show that one set of assignment constraints can be dropped from P1 by forcing $y_{i1} = x_{i1}$ and $y_{ii} = nx_{ii}$ for $i=2, \ldots, n$. 
3. Formulations of Transportation Scheduling Problems

The formulation given in Section 2 can be used as a basis for formulating a variety of transportation scheduling problems. In this section, we present these problems and their formulations.

3.1 The Multi-Traveling Salesman Problem

The traveling salesman problem as formulated by Miller, et. al. [31] was extended by Gavish [12] to the multi-traveling salesman problem. For this problem, we have to find \( M \) tours (one for each salesman) such that each tour originates and ends at the depot at node 0. Each node \((1,2,\ldots,n)\) is visited exactly once, and total travel costs are minimized. Based on the formulation given in Problem P1, the formulation to the multi-traveling salesman problem is:

Find variables \( x_{ij}, y_{ij} \ i,j=0,1,2,\ldots,n \) that minimizes:

\[
Z = \sum_{i=0}^{n} \sum_{j=0}^{n} C_{ij}x_{ij} \tag{13}
\]

subject to:

\[
\sum_{i=0}^{n} x_{ij} = 1 \quad j=1,2,\ldots,n \tag{14}
\]

\[
\sum_{j=0}^{n} x_{ij} = 1 \quad i=1,2,\ldots,n \tag{15}
\]

\[
\sum_{i=0}^{n} x_{i0} = M \tag{16}
\]
\[
\sum_{j=0}^{n} X_{0j} = M
\]  
(17)

\[
\sum_{j=0}^{n} y_{ij} - \sum_{j=0}^{n} y_{ji} = 1
\]  
\[i=1,2,...,n\]
(18)

\[
y_{ij} \leq (n-M+1)X_{ij}
\]  
\[i,j=0,1,2,...,n\]
(19)

\[
X_{ij} = 0,1, y_{ij} \geq 0
\]  
\[\forall i,j\]
(20)

By adding the equality \(X_{0j} = y_{0j}\)  \(\forall j = 1,2,...,n\), we assure that the \(y_{ij}\) values will also determine the arc locations within its tour. Moreover, if we require a certain load balancing, we need only replace the corresponding constraints in (19) with

\[
X_{i0L} \leq y_{i0} \leq X_{i0U}
\]  
\[i=1,2,...,n\]
(21)

where \(L\) and \(U\) are the lower and upper bounds on the number of cities visited by a salesman.

3.2 The Delivery Problem

This problem is described (see references [7], [9], [11], [30]) as follows. Given \(M\) trucks and a non-negative load \(d_i\), \(i=1,2,...,n\) associated with each node \(i\), find \(M\) tours of minimum total cost that leave a depot 0, visit each node only once, and return to the depot. At each node \(j\) the truck is loaded (or unloaded) by the amount \(d_j\). There is a limit \(Q\) on truck capacity such that the amount collected in each tour cannot exceed this limit. This problem can be formulated as follows:
Find variables $x_{ij}, y_{ij}$ $i,j=0,1,2,...,n$ which minimize:

$$Z = \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ij} x_{ij}$$

subject to (14-17), and

$$\sum_{j=0}^{n} y_{ij} - \sum_{j=0}^{n} y_{ji} = d_i \quad i=1,2,...,n$$

(23)

$$y_{ij} \leq Q x_{ij} \quad i,j=0,1,2,...,n$$

(24)

$$x_{ij} = 0,1 \quad y_{ij} \geq 0 \quad \forall i,j$$

(25)

The constraints (14-17) ensure that the $x_{ij}$ values form tours, while the constraints (23-24) ensure that all tours contain the depot (node 0) and do not exceed the truck capacity.

Another extension of the delivery problem is the case in which the number of trucks is not given beforehand, and there is an extra fixed cost $P$ associated with each additional truck used for delivery. This case may be formulated as:

$$\text{Min} \left\{ \sum_{i=1}^{n} \sum_{j=0}^{n} c_{ij} x_{ij} + \sum_{j=0}^{n} (C_{0j}+P)x_{0j} \right\}$$

(26)

subject to (14-15), (23-25), and:

$$\sum_{j=1}^{n} x_{0j} - \sum_{j=1}^{n} x_{j0} = 0$$

(27)
3.3 The Multi-Terminal Delivery Problem

The multi-terminal delivery problem is an extension of the delivery problem in which we have K depots which may be used as starting points for tours. There exists an extra restriction that a tour must always return to the same depot from which it started. Different types of trucks may be used for performing the deliveries. Truck type $h$ has a capacity $Q_h$, and a fixed cost $P_{kh}$ for using truck type $h$ from the $k$th depot. There exists a limit $M_{kh}$ on the number of trucks of type $h$ which may originate from the $k$th depot; $C_{ijh}$ is the travel cost from node $i$ to node $j$ using truck type $h$. We assume that a node is serviced by just one truck.

Let $H$ be the index set of truck types (i.e., $h \in H$), and assume the following indexing scheme: the depots are indexed as $i=1,2,\ldots,K$, while the nodes are indexed as $i=K+1,\ldots,K+n$.

The multi-terminal delivery problem is formulated as follows:

Find variables $x_{ij}$, $y_{ij}$, $i,j=0,1,\ldots,n$ that minimize:

$$
2 = \left\{ \sum_{i=K+1}^{K+n} \sum_{j=1}^{K+n} C_{ijh} x_{ijh} + \sum_{i=1}^{K} \sum_{j=K+1}^{K+n} \sum_{h \in H} (C_{ijh} + P_{ih}) x_{ijh} \right\}
$$

subject to:

$$
\sum_{i=1}^{K+n} \sum_{h \in H} x_{ijh} = 1 \quad j=K+1,\ldots,K+n \quad (28)
$$

$$
\sum_{j=1}^{K+n} x_{ijh} - \sum_{j=1}^{K+n} x_{jih} = 0 \quad \forall h \in H, i=1,2,\ldots,K+n \quad (29)
$$

$$
\sum_{j=K+1}^{K+n} x_{ijh} \leq M_{ih} \quad \forall h \in H, i=1,2,\ldots,K \quad (30)
$$
\[ \sum_{j=1}^{K+n} y_{ij} - \sum_{j=i}^{K+n} y_{ji} = d_i \quad i=K+1, \ldots, K+n \quad (31) \]

\[ y_{ij} \leq \sum_{h \in H} Q_h x_{ijh} \quad i=K+1, \ldots, K+n \quad j=1,2, \ldots, K+n \quad (32) \]

\[ x_{ijh} = 0,1 \quad y_{ij} \geq 0 \quad \forall i,j,h \quad (33) \]

This formulation is more complicated than that for the simple delivery problem due to the constraints on the \(X\) variables. Here, we need standard multi-commodity network flow constraints (29) rather than the assignment constraints, to ensure that if a truck of a given type enters a city, the same truck will also leave the city. (30) limits the number of trucks type \(h\) that may originate at the \(i^{th}\) depot. A similar formulation has been used by Nambier et al. [32] to formulate the natural rubber collection problem in Malasia, which includes decisions such as plant location, and truck routing and scheduling.

3.4 The Dial-a-Bus Problem

The dial-a-bus problem arises in the following situation. A bus driver who is initially located at location 0, is given a set of \(n\) deliveries to perform. Each delivery \(i\) consists of two locations, \(a_i\) and \(b_i\). Location \(b_i\) can be visited only after location \(a_i\) has been visited. The objective is to find a feasible tour over the \(n\) pairs of locations \((a_i, b_i)\) which minimizes total travel cost.

This problem arises in a dial-a-bus system in which passenger \(i\) requests to be picked up at location \(a_i\) and delivered to location \(b_i\). It may also characterize a delivery air-service or cargo ship service which has to schedule airport landings or port visits in order to satisfy all deliveries.
without violating the loading/unloading constraints.

Using the indexing scheme such that all loading points are numbered from \{1,2,...,n\} and all unloading points from \{n+1,...,2n\}, where delivery \( i \) is loaded at point \( i \) and unloaded at point \( n+i \), the dial-a-bus problem is formulated as:

Find variables \( x_{ij}, y_{ij} \), \( i,j=0,1,...,2n \) that minimize:

\[
Z = \left\{ \sum_{i=0}^{2n} \sum_{j=0}^{2n} c_{ij} x_{ij} \right\}
\]

subject to:

\[
\sum_{i=0}^{2n} x_{ij} = 1 \quad j=0,1,...,2n \tag{34}
\]

\[
\sum_{j=0}^{2n} x_{ij} = 1 \quad i=0,1,...,2n \tag{35}
\]

\[
\sum_{j=0}^{2n} y_{ij} - \sum_{j=1}^{2n} y_{ji} = 1 \quad i=1,2,...,2n \tag{36}
\]

\[
y_{ij} \leq 2(n+1)x_{ij} \quad \forall i,j \tag{37}
\]

\[
\sum_{j=0}^{2n} y_{n+i,j} \geq \sum_{j=1}^{2n} y_{ij} + 1 \quad i=1,2,...,n \tag{38}
\]

\[
x_{ij} = 0,1, \quad y_{ij} \geq 0 \quad \forall i,j \tag{39}
\]

The constraints in (38) are needed to assure that node \( n+i \) will be visited only after node \( i \) has been visited.

This simplified formulation of the dial-a-bus problem has been extended by Gavish and Srikanth [15] to handle multiple buses and due dates/times specified by the passengers.
3.5 The School Bus Problem

School buses which are initially located at school (node 0), have to collect students waiting at n pick-up points (nodes 1, 2, ..., n), and deliver them to school. The capacity of each bus is limited to Q students. The number of students waiting at the ith pick-up point is equal to \( d_i \), \( 0 \leq d_i \leq Q \), \( i=1, 2, ..., n \). \( t_{ij} \) is the travel time from pick-up point i to point j. Security and operational considerations limit the time that students at pick-up point i are allowed to spend on the bus to \( T_i \) time units, \( t_{i0} \leq T_i \ \forall i \). Only one bus is allowed to stop at pick-up point.

Let \( P \) be the cost of using an extra bus for the schedule, \( C_{ij} \) be the cost to travel from point i to j, \( X_{ij} \) be a binary variable denoting travel from point i to point j, \( y_{ij} \) be the number of students on the bus between points i and j, while \( z_{ij} \) is the travel time from point i to the school assuming that the next bus stop is at point j. The school bus problem can be formulated as:

Find variables \( X_{ij}, y_{ij}, z_{ij} \), \( i,j=0,1,2, ..., n \) that minimize:

\[
Z = \sum_{i=1}^{n} \sum_{j=0}^{n} C_{ij} X_{ij} + \sum_{j=0}^{n} (C_{0j} + P) X_{0j}
\] (40)

subject to:

\[
\sum_{i=0}^{j} X_{ij} = 1 \quad j=1, 2, ..., n
\] (41)

\[
\sum_{j=0}^{n} X_{ij} = 1 \quad i=1, 2, ..., n
\] (42)
\[
\sum_{j=0}^{n} y_{ij} - \sum_{j=1}^{n} y_{ji} = d_i \quad i=1,2,...,n
\]

\[
y_{ij} \leq Q x_{ij} \quad i,j=0,1,2,...,n
\]

\[
\sum_{k=0}^{n} z_{ki} = \sum_{j=0}^{n} z_{ij} = \sum_{k=0}^{n} t_{ki} x_{ki} \quad i=1,2,...,n
\]

\[
z_{ij} \leq T_i x_{ij} \quad i,j=0,1,2,...,n
\]

\[
x_{ij} = 0,1 \quad z_{ij}, y_{ij} \geq 0 \quad \forall i,j
\]

The constraints in (41-42) ensure that each pick-up point is visited by exactly one bus, (43-44) prevent subtour formation and limit the number of students in the bus to the bus capacity, while (45-46) ensure that the routes meet the travel time constraints.
4. **Relationship to Previous Formulations**

In this section we investigate the relationship between the new formulation and previous formulations for the traveling salesman problem. First we show a dual relationship between the Miller, et. al. [31] formulation and Problem P1; second, by applying Benders Decomposition [6] to Problem P1, we obtain cuts similar to the tour-breaking constraints suggested by Dantzig, et. al. [8].

4.1 **Duality Between Problem P0 and Problem P1**

Without loss in generality we may rewrite Problem P0 given by (1-5) as

\[
\begin{align*}
\text{Min} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}x_{ij} + \text{Min} \left\{ \sum_{i=1}^{n} U_i \right\} \right\} \\
\text{subject to (2), (3) and:} \\
U_i - U_j + n \cdot x_{ij} \leq n - 1 & \quad i=1,2,...,n \quad j=2,...,n, \ i \neq j \\
x_{ij} = 0,1, \ U_i \geq 0 & \quad \forall i,j 
\end{align*}
\] (48)

The optimal value for the inner-minimization problem is obtained for

\[U_{i_k} = k, \ k=0,1,2,...,n-1\] where \(i_0 = 1, i_1, i_2, ..., i_{n-1}, i_n = 1\) is any feasible tour. Thus, the optimal value of the inner-minimization proboem is equal to a constant \(n(n-1)/2\) and does not have any effect on the optimization over the assignment variables.

The dual problem to the inner-minimization problem is given by:

\[
\begin{align*}
\text{Max} \left\{ \sum_{i=1}^{n} \sum_{j=2}^{n} y_{ij}(n \cdot x_{ij} - n+1) \right\} \\
\text{subject to (2), (3) and:} \\
\end{align*}
\] (51)
subject to:

$$\sum_{j=1}^{n} y_{ji} - \sum_{j=2}^{n} y_{ij} \leq 1 \quad \forall i=2,\ldots,n$$ \hspace{1cm} (52)

$$\sum_{j=2}^{n} y_{ij} \leq 1$$ \hspace{1cm} (53)

$$y_{ij} \geq 0 \quad \forall i=1,\ldots,n \quad j=2,\ldots,n, j \neq i$$ \hspace{1cm} (54)

Due to (54) the constraint (53) is redundant and therefore is eliminated from further consideration.

Consider any feasible extreme point for the assignment constraints (2), (3); it is easy to show that (51-54) is unbounded unless the extreme point is a feasible tour. Hence, we can restrict our attention to those extreme points which form a feasible tour. For any feasible tour \(\{i_0 = 1, i_1, i_2, \ldots, i_{n-1}, i_n = 1\}\), an optimal solution for (51-54) is \(\tilde{y}_{i_{j-1}i_j} = n-j\) for \(j=1,\ldots,n\), and \(\tilde{y}_{ij} = 0\) otherwise. It is easily seen that \(\tilde{y}_{ij}\) is feasible.

To show optimality, note that

$$\tilde{y}_{ij} - (n X_{ij} - n+1) = \begin{cases} \tilde{y}_{ij} & \text{for } X_{ij} = 1 \\ 0 = \tilde{y}_{ij} & \text{for } X_{ij} = 0 \end{cases}$$ \hspace{1cm} (55)

The objective function (51) may now be rewritten as

$$\sum_{i=1}^{n} \sum_{j=2}^{n} \tilde{y}_{ij} (n X_{ij} - n+1) = \sum_{i=1}^{n} \sum_{j=2}^{n} \tilde{y}_{ij} = \frac{n(n-1)}{2}$$
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By weak-duality, \( \{ \bar{y}_{ij} \} \) is optimal since its objective value equals that of its dual.

Now, (51-54) is equivalent to

\[
\text{Max} \quad \sum_{i=1}^{n} \sum_{j=2}^{n} y_{ij}
\]

subject to (52), (54) and

\[
y_{ij} \leq S x_{ij}
\]

where \( S \geq n-1 \). Replacing the inner-minimization problem in (48-50) with its dual equivalent (56), (52), (54), and (57), we have a problem nearly identical to \( P_l \). Since the inner problem is a maximization problem, the constraint set (52) may be replaced with equality constraints to yield a problem identical to \textit{Problem P}_l.

4.2 Application of Benders Decomposition to Problems \( P_l \)

In this section we apply Benders decomposition [6] to the formulation given by \textit{Problem P}_l. Given values for the assignment variables \( x_{ij} \) (i.e., feasible in (7-8)), the subproblem (SP1) is as follows:

\[
\text{(SP1)} \quad \min \left\{ \sum_{i=1}^{n} \sum_{j=2}^{n} 0 \cdot y_{ij} \right\}
\]

subject to:

\[
\sum_{j=1}^{n} y_{ji} - \sum_{j \neq i}^{n} y_{ij} = 1 \quad i=2, \ldots, n
\]
0 \leq \gamma_{ij} \leq (n-1)x_{ij} \quad i=1, \ldots, n \\
\quad j=2, \ldots, n, \ i \neq j \quad (60)

The master problem for this decomposition is the standard assignment problem given by (6-8) supplemented by the set of cuts generated by (SPI). It will be shown that the generated constraints are just the subtour breaking constraints identified by Dantzig, Fulkerson, and Johnson [8].

The dual of SPI is:

(DSPI) \quad \max \sum_{i=1}^{n} \sum_{j=2}^{n} (n-1)x_{ij}y_{ij} + \sum_{i=2}^{n} \mu_{i} \quad (61)

subject to:

- \gamma_{ij} + \nu_{j} - \mu_{i} \leq 0 \quad i=2, \ldots, n \quad j=2, \ldots, n, \ i \neq j \quad (62)

- \gamma_{ij} + \nu_{j} \leq 0 \quad j=2, \ldots, n \quad (63)

\gamma_{ij} \geq 0 \quad \forall \ i, j \quad (64)

The solution of the dual problem (and hence, the Benders cut) depends upon the given assignment variables \{x_{ij}\}. Since \{x_{ij}\} must satisfy the assignment constraints (7-8), the values must identify a set of disjoint subtours (loops). The set of nodes can be divided into two sets \(N_1, N_2\) depending on whether the node's subtour contains the depot (node 1). That is, \(i \in N_1\) if node 1 is contained in the subtour which includes node 1; otherwise, \(i \in N_2\) and its subtour does not contain the depot.

If \(N_2\) is empty, \(\{x_{ij}\}\) defines a feasible tour, and an optimal solution to (DSPI) is \(\gamma_{ij} = \nu_{i} = \nu_{j} = 0\) for all \(i, j\). Provided that \(\{x_{ij}\}\) solves the
current master problem, then it is an optimal solution to the traveling salesman problem.

If $N_2$ is not empty, (DSPL) is unbounded and generates a constraint for the master problem. An extreme ray for (BSPL) is given by

$$
u_i = \begin{cases} 
0, & \text{if } i \in N_1 \\
1, & \text{if } i \in N_2 
\end{cases} \quad (65)$$

$$
\gamma_{ij} = \left[ \mu_j - \mu_i \right]^+ 
\quad i=2,\ldots,n \\
\gamma_{ij} = \gamma_{ji} 
\quad j=2,\ldots,n \quad i \neq j \quad (66)
$$

The Benders cut generated by this ray is

$$
ge - \sum_{i=1}^{n} \sum_{j=2}^{n} (n-1)x_{ij} \gamma_{ij} + \sum_{i=2}^{n} \nu_i \ge 0 \quad (68)$$

Using (65-67), the constraint may be restated as

$$
ge \sum_{i \in N_2} \sum_{j \in N_1} x_{ij} \geq \frac{|N_2|}{n-1} \quad (69)$$

where $|N_2|$ is the cardinality of set $N_2$. Since $|N_2| \leq n-1$, and $x_{ij} = 0,1$, we have

$$
ge \sum_{i \in N_2} \sum_{j \in N_1} x_{ij} \geq 1 \quad (70)$$
But this is just the subtour-breaking constraint proposed in [8] which requires the use of at least one arc going from set $N_2$ to $N_1$. Hence, the application of Benders method to formulation P1 is identical to starting with the assignment constraints and sequentially generating subtour breaking constraints as given in (70). Clearly, this application offers no new computational breakthroughs.

In [13] we apply the Benders method to the second, more compact formulation of the traveling salesman problem. In this instance, the solution to the master problem again divides the node set $N$ into two sets $N_1$, $N_2$ such that $i \in N$, if node $i$ is on a directed path connected to the depot. The subproblem then generates a cut which is identical in form to (70) requiring at least one arc to connect set $N_2$ to set $N_1$.

A similar procedure has been used by Gavish for identifying new valid inequalities for the capacitated minimal directed tree problem (CMDT) [17] and for the capacitated minimal spanning tree problem (CMST) [18]. Those constraints were found valuable in an augmented Lagrangean procedure for generating tight bounds for those problems. The CMST and CMDT problems are known to be open tour versions of the delivery problem. Studies are now in progress for applying similar procedures to the delivery problem.
5. Lagrangean Relaxations of the Traveling Salesman Problem

The formulations given in the previous sections suggest several types of Lagrangean relaxations. These relaxations are applicable not only to all problems that were formulated in Section 3, but also to problems in related fields such as topological design of computer networks, sales, service and political territory reconfiguration or redistricting, or design of bus routes in transportation systems. We demonstrate the relaxations for the traveling salesman problem and point, wherever applicable, to past or ongoing research in which similar methods have and are being applied to other problems in that class. The use of Lagrangean relaxation methods for solving integer programming problems is well documented. Fisher [10] gives an excellent review of such approaches in general, while Magnanti [30] provides a nice perspective on these approaches for vehicle routing problems.

5.1 The l-Arborescence Relaxation

One of the most successful algorithms for solving the symmetric traveling salesman problem is that of Held and Karp [25-26]. The algorithm uses a Lagrangean relaxation technique [22], [33] combined with a subgradient optimization procedure for obtaining tight lower bounds on the objective function value, and a branch and bound procedure for closing the integer gap in cases that such a gap was detected. Later modifications to this basic procedure are due to Held, Wolfe, and Crowder [27] Hansen and Krarup [24] and Smith and Thompson [34]. The computational experience gained in those experiments reveals that the bounds obtained by those relaxation procedures are tight and that the depth and number of nodes generated by the branch and bound procedure is quite limited (less than one thousand in problems of up to one hundred cities).
The 1-arborescence relaxation is an extension to the Held-Karp relaxation and has the advantage of being applicable to the asymmetric problem. The 1-arborescence relaxation is derived by adding to Problem PI (given in 6.11) the redundant set of constraints:

\[ x_{1j} + x_{j1} \leq 1 \quad j=2,\ldots,n. \] (71)

This rules out simple cycles consisting of node 1 and node j. Multiplying the constraints in (8) by a vector of Lagrange multipliers \( \psi = \{\psi_i, i=2,\ldots,n\} \) and adding them to the objective function, we obtain the following Lagrangean problem

\[ L_1(\psi) = \min \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i=2}^{n} \psi_i (1 - \sum_{j=1}^{n} x_{ij}) \right\} \] (72)

subject to: (6-7), (9-11), (71).

For a fixed vector \( \psi \), solving this problem is equivalent to finding the minimum cost 1-arborescence with root node 1. A 1-arborescence consists of a directed spanning tree rooted at node 1 together with a directed arc leading into node 1. Efficient algorithms exist ([29]) for finding the minimum cost directed spanning tree. The addition of the constraint set in (71) rules out 1-arborescences with simple cycles consisting only of node j and node 1. The additional complexity due to this set of constraints is at most double that needed to solve a minimal cost arborescence problem. However, our computational experience indicates that it is only
marginally more difficult.

For $Z_{IP}$ being the optimal objective function value for Problem P1, it is well known [22] that for all $\psi$

$$Z_{IP} \geq L_1(\psi).$$

If we define the vector $\psi^*$ such that

$$L_1(\psi^*) = \max_{\psi} L_1(\psi),$$  \hspace{1cm} (73)

then the different bounds satisfy the following relation [22]

$$L_{IP} \geq L_1(\psi^*) \geq \tilde{Z}_{LP}$$

where $\tilde{Z}_{LP}$ is the optimal linear programming relaxation of the problem given by Problem P1 with the additional constraints in (71). Good approximations to $\psi^*$ can be computed using a dual ascent or subgradient optimization procedure. Smith [35] has applied a similar dual ascent procedure to the Asymmetric Travelling Salesman problem and has generated tight bounds. Gavish [16] has successfully applied a similar procedure to degree constrained minimal spanning trees that appear in the topological design of centralized computer communication networks. A similar procedure has been used in [18] to obtain optimal solutions to Capacitated Minimal Spanning Tree problems.
5.2 An Assignment Problem Relaxation

Assignment problems have been used as relaxations of the traveling salesman problem by Bellmore and Malone [4] and Balas and Christofides [2], and for the delivery problem by Gavish and Shlifer [14]. The new formulation leads to new assignment-based relaxation of the traveling salesman problem, suggested by Graves and Magnanti [23].

Using a vector of Lagrange multipliers $\pi = \{\pi_2, \pi_3, \ldots, \pi_n\}$ for the constraints (9) and adding them to the objective function, we form the following Lagrangean:

$$L_2(\pi) = \text{Min}\{ \sum_{i=2}^{n} \pi_i + \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij} + \sum_{i=1}^{n} \sum_{j=2}^{n} Y_{ij} \beta_{ij} \}$$

(74)

where

$$\beta_{ij} = \begin{cases} 
(\pi_i - \pi_j) & \text{for } i, j = 2, \ldots, n \\
-\pi_j & \text{for } i = 1, j = 2, \ldots, n 
\end{cases}$$

The problem given in (74) is a large mixed integer problem which seems to be difficult to solve. However, it can be simplified by using the following observation:

Given the optimal values of $X_{ij}(X^*_i)$ for a fixed vector $\pi$, the optimal $y_{ij}$ values ($y^*_{ij}$) are given by the following relation:

$$y^*_{ij} = \begin{cases} 
0 & \text{if } X^*_i = 0 \\
0 & \text{if } X^*_i = 1 \text{ and } \beta_{ij} > 0 \\
-1 & \text{if } X^*_i = 1 \text{ and } \beta_{ij} < 0 \\
b & \text{if } X^*_i = 1 \text{ and } \beta_{ij} = 0 
\end{cases}$$

(75)

where $0 \leq b \leq n-1$. We can now write
\[ y_{ij}^* \beta_{ij} = x_{ij}^* \gamma_{ij}. \] (76)

where
\[
\gamma_{ij} = \begin{cases} 
\beta_{ij}(n-1) & \text{if } \beta_{ij} < 0 \\
0 & \text{if } \beta_{ij} \geq 0 
\end{cases}
\]

Thus, by substituting (76) into (74) we obtain a Lagrangean problem which involves the \( x_{ij} \) variables only:
\[
L_2(\pi) = \text{Min} \left\{ \sum_{i=2}^{n} \tau_i + \sum_{i=2}^{n} \sum_{j=1, j \neq i}^{n} \tilde{C}_{ij} x_{ij} \right\}
\]
(77)

subject to (7-8) and (11), where \( \tilde{C}_{ij} = C_{ij} + \gamma_{ij} \).

The problem given in (77) is a simple assignment problem. Once the \( x_{ij}^* \) values are given we can use the relations in (75) to obtain the \( y_{ij}^* \) values.

As before, if \( \pi^* \) maximizes \( L_2(\pi) \), we know that
\[
L_2(\pi^*) \leq Z_{IP}
\]

Moreover, since the constraints in (7-8) are totally unimodular,
\[
L_2(\pi^*) = Z_{LP}
\]

Thus, the best bound from this relaxation is equal to the value, \( Z_{LP} \), obtained from the linear programming relaxation of Problem P1. However, the solution of the dual problem suggested by the Lagrangean relaxation may be much easier than solving the LP relaxation via a simplex-based procedure. Gavish [17] uses such a procedure in order to obtain bounds for the
capacitated minimal directed tree problem. A subgradient optimization procedure was used for approximating the $\tau^*$ values. The procedure, when compared to solving the linear programming relaxation by a simplex-based procedure, was able to obtain in less than one percent of the computing effort, bounds that were at least as good as the bounds obtained by the simplex-based procedure. A similar procedure has been applied by Graves and Magnanti [23] to the delivery problem.

5.3 Shortest Path Based Relaxations

By multiplying the coupling constraints in (10) by a vector of Lagrange multipliers, $\pi = \{\pi_{ij}, i=1,2,...,n, j=2,...,n, i\neq j\}$, and adding them to the objective function, we obtain the following Lagrangean:

$$L_3(\pi) = \text{Min} \left\{ \sum_{i=1}^{n} \sum_{j=2}^{n} [C_{ij} + (n-1)\pi_{ij}]X_{ij} - \sum_{i=1}^{n} \sum_{j=2}^{n} \pi_{ij}Y_{ij} \right\}$$

subject to: (7-9), (11).

For a fixed vector $\pi$, this problem separates into two subproblems. The first is an assignment problem over the $X$ variables; and the second is a network flow problem over the $Y$ variables. Since the two subproblems are totally unimodular, it is clear that

$$L_3(\pi^*) = Z_{LP}$$

i.e., the best bound that we can hope to obtain is equal to the bound obtained from the linear programming relaxation of Problem $P_1$.

This relaxation, first suggested in [20], to the best of our knowledge, is new. It does not seem to be an attractive relaxation for the pure traveling salesman problem since the bound is no better than the LP relaxation. However, it might be an attractive relaxation for problems that involve additional
restrictions on tour configurations such as time or capacity restrictions. Work is now in progress for applying a similar procedure to such problems, including the delivery problem, the district reconfiguration problem, and to the Steiner tree problem over graphs.

5.4 Some Computational Experiences

In order to investigate the quality of the bounds, we have conducted a set of computational tests in which four methods for generating bounds to the traveling salesman problem have been examined. Two of the methods are LP relaxations, while the other two are based on a Lagrangean relaxation in which the integrality constraints are preserved.

The first LP relaxation is the solution to the assignment problem given by (6-8). The second LP relaxation is to solve the LP relaxation of Problem P1. The Lagrangean relaxations are based on the 1-arborescence formulation. The two methods which were tested differ in the selection of an initial vector $\psi$. The third method sets $\psi$ equal to the corresponding dual variables for (8) when the assignment problem (6-8) is solved; Held and Karp [25,26] have suggested this method in their 1-tree relaxation for the traveling salesman problem. The fourth method also uses dual variables for (8), but from the solution to the LP relaxation of P1. For both of these methods, we have applied a subgradient optimization procedure in order to investigate the potential improvement from an improved initial choice for $\psi$.

The computational tests were performed on four problems for which optimal solutions are known. The results are summarized in Table 1. The initial bounds obtained from the Lagrangean relaxations are superior to those obtained from solving the corresponding LP relaxation as expected. In addition, the initial bounds for the Lagrangean using the dual variables from the LP relaxation of P1 seem to dominate those for the Lagrangean with initial
Table 1 - Experiments With Different Bound Procedures

<table>
<thead>
<tr>
<th>Number of Cities</th>
<th>Optimal Solution to TSP</th>
<th>Problem Source</th>
<th>Bound from LP Solutions</th>
<th>Bounds for Lagrangean Relaxation, and Subgradient Optimization Procedure</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Assignment Problem</td>
<td>Initial Multipliers</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>LP Relaxation of P1</td>
<td>Assignment Problem</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Initial Value</td>
</tr>
<tr>
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<td>148</td>
<td>[18]</td>
<td>140</td>
<td>140</td>
</tr>
<tr>
<td>33</td>
<td>10861</td>
<td>[18]</td>
<td>9948</td>
<td>10155</td>
</tr>
<tr>
<td>42</td>
<td>12345</td>
<td>[6]</td>
<td>9217</td>
<td>10458</td>
</tr>
</tbody>
</table>
multipliers taken from the solution of the assignment problem. However, the final value for the bounds and the rate of convergence, as given by the number of iterations needed by the subgradient procedure, seems to be relatively insensitive to the initial multiplier values.

The computational tests were limited to 42 cities due to the excessive computer time needed to solve the large linear program. However, even these limited tests clearly demonstrate that the bounds obtained by a Lagrangean relaxation in which the integrality constraints are preserved, clearly dominate the bounds obtained by solving the linear program relaxation.
6. **Enhancements and Future Research**

In this paper we have presented new formulations to a variety of scheduling and routing problems in transportation and distribution systems. All of the problems that were formulated belong to the NP-complete class and there is very little hope for developing exact algorithms for those or related problems that have a polynomial time complexity. Nevertheless, very good branch and bound procedures have been developed for the traveling salesman problem ([26], [24], [34], [35]) and for the multiple traveling salesman problem [19]. All of these procedures rely on clever bounding procedures based on Lagrangean relaxations. The formulations given in this paper suggest new types of relaxations which are applicable not only to the traveling salesman problem, but also to a variety of transportation routing problems. Current research efforts are now exploring these ramifications of the formulations.

A different research direction is to use the same framework in order to come up with tighter linear programming formulations of the problem. Wong [36] has extended Problem P1 to a multicomodity flow subproblem whose linear programming relaxation is tighter than the linear programming relaxation of Problem P1. Gavish [20] developed a significantly tighter formulation which in preliminary computational tests generated the optimal integer solution for known difficult traveling salesmen problems.

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