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MATHEMATICAL FOUNDATIONS
OF CONSTRUCTIVE SOLID GEOMETRY:
GENERAL TOPOLOGY OF CLOSED REGULAR SETS

by

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SUMMARY

Constructive solid geometry is a scheme for modelling solid objects as set-theoretical compositions of primitive solid "building blocks". This memo discusses results of general (point-set) topology which provide rigorous theoretical foundations for the constructive solid geometry methodology.
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1. INTRODUCTION

Constructive solid geometry is a scheme for modelling rigid solid objects as set-theoretical compositions of primitive solid "building blocks" [1,2]. Constructive solid geometry is based largely on modern Euclidean geometry and general (i.e. point-set) topology of subsets of E3 (Euclidean three-dimensional space). This Memo discusses certain elementary but specialized results of point-set topology which we have found useful in the study of geometric modelling. The Memo is a reference document; it is written tersely and does not attempt to motivate fully or to discuss the application of the concepts and results described, because such issues are addressed elsewhere [1,2,7,8]. Proofs of results not available in the cited references are included.

We assume that readers have some knowledge of undergraduate topology at the level of the first three chapters of [3] or [4], but large portions of Section 2 may be viewed as a "refresher course". Our notational conventions are introduced as the need arises.

* * *
2. BASIC NOTIONS OF POINT-SET TOPOLOGY

2.1 SET ALGEBRA

We assume that readers are familiar with the notions of set and set operators. If \( X \) and \( Y \) are subsets of a "universal" set \( W \) we denote their union by \( X \cup Y \), their intersection by \( X \cap Y \), and their difference by \( X - Y \). The complement of \( X \) with respect to \( W \) is \( cX = W - X \). The empty set is denoted by \( \emptyset \). We also use the standard notations:

\[
A \times B \quad \text{"the cross (Cartesian) product of sets } A \text{ and } B"
\]

\[
f: A \rightarrow B \quad \"a function } f \text{ from set } A \text{ into set } B"
\]

Some algebraic properties of sets are listed below; \( X, Y, \) and \( Z \) are subsets of \( W \).

PROPERTY 2.1.1. Union and intersection are commutative:

\[
X \cup Y = Y \cup X,
X \cap Y = Y \cap X \quad [5].
\]

PROPERTY 2.1.2. Each of the operations union and intersection is distributive over the other:

\[
X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z),
X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z) \quad [5].
\]

PROPERTY 2.1.3. The empty set \( \emptyset \) and the universe \( W \) are identity elements for the union and intersection operators:

\[
X \cup \emptyset = X,
X \cap W = X \quad [5].
\]

PROPERTY 2.1.4. The complement satisfies:

\[
X \cup cX = W,
X \cap cX = \emptyset \quad [5].
\]

The properties above characterize an important class of algebraic systems.

DEFINITION 2.1.5. A set of elements \( X, Y, \ldots \) plus three operations \( \cup, \cap, \) and \( c \), which satisfy the algebraic Properties 2.1.1 through 2.1.4 is called a BOOLEAN ALGEBRA [5].

* 

Sets (and Boolean algebras) have many other algebraic properties which we will use freely in the sequel. Here we
shall mention an additional property, because it is used often.

PROPERTY 2.1.6. (De Morgan's Laws) [5]:

\[ c(X \cup Y) = cX \& cY, \]
\[ c(X \& Y) = cX \cup cY, \]
\[ c(X - Y) = cX \cup Y. \]

*

PROPERTY 2.1.7. Any class of subsets of \( W \) which is closed under the \( \cup, \& \), and \( c \) operations is a Boolean algebra.

PROOF: Clearly the defining Properties 2.1.2 through 2.1.4 are satisfied provided that unions, intersections, and complements of sets in the class also belong to the class. Q.E.D.

*

2.2 METRIC SPACES

DEFINITION 2.2.1. A METRIC SPACE is a pair \( (W, d) \) where \( W \) is a set and \( d: W \times W \rightarrow \mathbb{R} \) is a function called the "distance" or the "metric", such that for all \( x, y, \) and \( z \) in \( W \):

1) \( d(x, y) \geq 0 \);
2) \( d(x, y) = 0 \) if and only if \( x = y \);
3) \( d(x, y) = d(y, x) \);
4) \( d(x, z) \leq d(x, y) + d(y, z) \) (triangle inequality) [3].

*

An important example of a metric space is the Euclidean space of analytic geometry with its usual distance.

When the metric \( d \) is clear from the context we shall often refer to \( W \) as a metric space.

*

DEFINITION 2.2.2. Let \( (W, d) \) be a metric space, and \( x \) a point of \( W \). The OPEN BALL of radius \( R > 0 \) about \( x \), denoted \( B(x; R) \), is the set of all points \( y \) in \( W \) which satisfy \( d(x, y) < R \) [3].

DEFINITION 2.2.3. A subset \( X \) of a metric space is OPEN if it contains an open ball about each of its points [4].

*
Let \( W \) be the real line with its usual distance. An open ball about a real \( x \) is an "open interval" of length \( 2R \) about \( x \). Open intervals and unions of such are open sets.

* 

Open sets in a metric space have the following properties.

PROPERTY 2.2.4. The empty set \( \emptyset \) and the universe \( W \) are open [3].

PROPERTY 2.2.5. The intersection of a finite number of open sets is an open set [3].

PROPERTY 2.2.6. The union of any collection of open sets is an open set [3].

* 

The intersection of an infinite sequence of open sets need not be open; for example the intersection of the open intervals \((-1,1), (-1/2,1/2), (-1/3,1/3), \ldots, (-1/n,1/n), \ldots \) is the set consisting of the single point \( 0 \), and is not open.

* 

DEFINITION 2.2.7. A subset \( X \) of \((W,d)\) is BOUNDED if it is a subset of a ball of finite radius.

* * 

2.3 TOPOLOGICAL SPACES

Topological spaces may be viewed as a generalization of metric spaces in which the notion of "nearness" is introduced in an abstract setting which does not require the existence of a distance. Although we are primarily interested in metric (Euclidean) spaces, we choose to work with topological spaces because the results we need are not harder to prove in the general setting of topological spaces than in the more restricted case of metric spaces.

DEFINITION 2.3.1. A topological space is a pair \((W,T)\) where \( W \) is a set and \( T \) -- called the TOPOLOGY -- is a class of subsets of \( W \) called OPEN SETS and satisfying properties 2.2.4 through 2.2.6 [3].

*
Since the (metrically defined per 2.2.3) open sets of a metric space have the Properties 2.2.4 through 2.2.6 it is clear that one can choose the set of all (metric) open sets of a metric space as a topology for the space -- called the "natural" or "usual" topology. This makes the metric space into a topological space whose open sets (in the sense of 2.3.1) are the same open sets defined in 2.2.3. Usually it is possible to associate several topologies besides the "natural" one to a metric space thereby converting it into several distinct topological spaces. While any metric space may be made into a topological space, the converse is not true: there are topological spaces -- called non-metrizable -- which cannot be put in correspondence with any metric space.

* *

DEFINITION 2.3.2. A NEIGHBORHOOD $N(x)$ of a point $x$ in a topological space $(W,T)$ is any subset of $W$ which contains an open set which contains $x$. If $N(x)$ is an open set it is called an OPEN NEIGHBORHOOD [3].

* *

2.4 CLOSED SETS

DEFINITION 2.4.1. A subset $X$ of a topological space $(W,T)$ is CLOSED if its complement is open, i.e. if $cX$ belongs to $T$ [3].

* *

Note that closed sets are not the "opposite" of open sets. In fact there are sets which are both closed and open (e.g. the universe $W$ and the null set $\emptyset$) and others which are neither closed nor open. As an example of the latter consider the set of rationals on the real line with the usual (metrically induced) topology.

Closed sets have the following properties, which are the "duals" of Properties 2.2.4 through 2.2.6.

PROPERTY 2.4.2. The empty set $\emptyset$ and the universe $W$ are closed [3].

PROPERTY 2.4.3. The union of a finite number of closed sets is a closed set [3].

PROPERTY 2.4.4. The intersection of any collection of closed sets is a closed set [3].

* *
2.5 CLOSURE

DEFINITION 2.5.1. A point $x$ is a LIMIT POINT of a subset $X$ of a topological space $(W,T)$ if each neighborhood of $x$ contains at least a point of $X$ different from $x$ [4].

* Limit points of a set are not necessarily elements of the set. Thus, for example, $\emptyset$ is a limit point of the open interval $(0,1)$ but is not a point of the interval. However:

PROPERTY 2.5.2. A set is closed if and only if it contains all its limit points [4].

* DEFINITION 2.5.3. The CLOSURE of a subset $X$, denoted $\overline{X}$, is the union of $X$ with the set of all its limit points [4].

* The operation of closure has the properties listed below. The first two of these properties may be used as alternative definitions of closure.

PROPERTY 2.5.4. If $x$ is a point in $\overline{X}$ each neighborhood of $x$ intersects $X$ [3].

PROPERTY 2.5.5. The closure of $X$ is the smallest closed set containing $X$ (i.e. the intersection of all closed sets containing $X$) [3].

PROPERTY 2.5.6. $X$ is closed if and only if $X = \overline{X}$ [3].

PROPERTY 2.5.7. If $X$ is a subset of $Y$ then $\overline{X}$ is a subset of $\overline{Y}$ [6].

PROPERTY 2.5.8. $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$ [3].

PROPERTY 2.5.9. The $\overline{X \cap Y}$ is a subset of $\overline{X} \cap \overline{Y}$ [6].

* The interesting fact about the last property above is that $\overline{X \cap Y}$ may be strictly smaller than $\overline{X} \cap \overline{Y}$. For example, let $X$ and $Y$ be the real intervals $(0,1)$ and $(1,2)$; while $\overline{X \cap Y} = \emptyset$, $\overline{X} \cap \overline{Y}$ is the set consisting of the point 1.

* *
2.6 INTERIOR

DEFINITION 2.6.1. A point $x$ of $W$ is an INTERIOR POINT of a subset $X$ of $W$ if $X$ is a neighborhood of $x$, i.e. if $X$ contains an open set which contains $x$ [3].

DEFINITION 2.6.2. The interior of a subset $X$ of $W$, denoted $iX$, is the set of all the interior points of $X$ [3].

PROPERTY 2.6.3. The interior of $X$ is the largest open set contained in $X$ (i.e. the union of all open sets contained in $X$) [3].

PROPERTY 2.6.4. $X = iX$ if and only if $X$ is open [4].

PROPERTY 2.6.5. If $X$ is a subset of $Y$ then $iX$ is a subset of $iY$ [6].

PROPERTY 2.6.6. $iX = ckcX$ [3]. (2-1)

PROPERTY 2.6.7. $cIX = kX$ [3].

PROPERTY 2.6.8. $ckX = icX$ [3].

PROPERTY 2.6.9. $i(X \cup Y) = iX \cup iY$ [6].

PROPERTY 2.6.10. $iX \cup iY$ is a subset of $i(X \cup Y)$ [6].

Note that in the last property above $i(X \cup Y)$ may be strictly larger than the union of the interiors. Consider, e.g. the two closed intervals on the real line $X = [0,1]$ and $Y = [1,2]$: $i(X \cup Y) = (0,2)$ while the union of the two interiors does not contain the point 1.

PROPERTY 2.6.11. $iX$ is a subset of $ikX$.

Property 2.6.11 follows from Property 2.6.5 since $X$ is a subset $kX$, but it is interesting to note that $iX$ may be strictly smaller than $ikX$, even when $X$ is an open set. Consider, for example, $X = (0,1) \cup (1,2)$; $iX = X$ but $ikX = (0,2)$.

----------

(2-1) The string of operators is shorthand for $c(k(cX))$. 
PROPERTY 2.6.12. \(1(X - Y) = 1X - kY\).

PROOF: By 2.6.9 \(1(X \& cY) = 1X \& icY\). The right hand side equals \(1X \& cY\) because of 2.6.8, and the conclusion follows. Q.E.D.

* *

2.7 BOUNDARY

DEFINITION 2.7.1. A point \(x\) of \(W\) is a BOUNDARY POINT of a subset \(X\) of \(W\) if each neighborhood of \(x\) intersects both \(X\) and \(cX\) [3].

DEFINITION 2.7.2. The boundary of \(X\), denoted \(bX\), is the set of all boundary points of \(X\) [3].

* *

The boundary has the following properties: the first property may be used as an alternative definition.

PROPERTY 2.7.3. \(bX = kX \& kcX\) [3].

PROPERTY 2.7.4. The \(bX\) is a closed set [3].

PROPERTY 2.7.5. \(bX = bcX\) [3].

PROPERTY 2.7.6. \(kX = 1X \cup bX = X \cup bX\).

PROOF: It is clear that \(1X \cup bX\) is a subset of \(kX\). To prove the reverse inclusion suppose that \(x\) is a point of \(kX\). If each neighborhood of \(x\) intersects both \(X\) and \(cX\) then \(x\) is a point of \(bX\). Otherwise two cases are possible. 1) There exists a neighborhood \(N(x)\) which does not intersect \(X\), and this contradicts the hypothesis that \(x\) belongs to \(kX\) by Property 2.5.4. 2) There exists a neighborhood \(N'(x)\) which does not intersect \(cX\) and therefore \(N'(x)\) is a subset of \(X\); this implies that \(x\) belongs to \(1X\) and to \(X\). Q.E.D.

PROPERTY 2.7.7. For any subset \(X\) of \(W\), \(W = 1X \cup bX \cup icX\), where the three sets in the right-hand side are pairwise disjoint [3].

PROPERTY 2.7.8. The \(b(X \cup Y)\) is a subset of \(bX \cup bY\).

PROOF: Let \(x\) be a point of \(b(X \cup Y)\). Then all neighborhoods \(N(x)\) must intersect both \(X \cup Y\) and \(c(X \cup Y) = cX \& cY\) (by DeMorgan's Law). For \(N(x) \& (X \cup Y)\) to be non-empty we must have either \(N(x) \& X\) non-empty or \(N(x)\)
8 Y non-empty (or both). Because N(x) \& cX \& cY is non-empty, it follows that either N(x) intersects both X and cX or it intersects both Y and cY, and therefore x is a boundary point of X or of Y. Q.E.D.

PROPERTY 2.7.9. The b(X \& Y) is a subset of bX U bY.

PROOF: Similar to the proof of Property 2.7.8.

PROPERTY 2.7.10. The b(X - Y) is a subset of bX U bY.

PROOF: Since X - Y = X \& cY, the result follows from Properties 2.7.9 and 2.7.5. Q.E.D.

PROPERTY 2.7.11. The i(X U Y) is a subset of iX U iY U (bX \& bY).

PROOF: Clearly i(X U Y) is a subset of X U Y, which is a subset of k(X U Y). By Properties 2.5.8 and 2.7.6 k(X U Y) = iX U iY U bX U bY. To complete the proof we show that bX U bY may be replaced by bX \& bY in the expression above. Let x be a point of bX but not of bY. Then either x belongs to iY, in which case we need not count it again, or else x belongs to iC. In the latter case x is an interior point of cY and therefore there exists a neighborhood N of x which is contained in cY, i.e. N \& cY = N. Because x is a boundary point of X, the neighborhood N must intersect cX, and therefore N \& cY \& cX is non-empty, which implies that N intersects c(X U Y). We can repeat this argument for any neighborhood N' of x contained in N. For an arbitrary neighborhood N" of x we can construct another neighborhood N" = N' \& N which is non-null and is a subset of N and of N". This shows that all neighborhoods of x must intersect c(X U Y), and therefore x cannot be an interior point of X U Y. We can repeat the reasoning for any y that belongs to bY but not to bX. We conclude that points in bX U bY but not in bX \& bY either are points of iX or iY or cannot be points of i(X U Y). Q.E.D.

PROPERTY 2.7.12. b(X U Y) = (bX \& iC) U (iC \& bY) U (bX \& bY) U k(cX \& cY).

PROOF: Firstly, by 2.7.8 b(X U Y) is included in bX U bY. The latter may be expanded by using 2.7.7 to yield (bX \& iY) U (bX \& bY) U (bX \& iC) U (bY \& iX) U (bY \& iC). We shall study the various terms in this union. Clearly (bX \& iY) U (bY \& iX) is a subset of iX U iY, and hence of i(X U Y) by 2.6.10; therefore these terms are disjoint from b(X U Y). Consider now the term (bX \& iC) U (bY \& iC); by writing the appropriate intersections in
full, it is easy to see that this term is disjoint from $icX \& icY$, $iX \cup iY$, and $bX \& bY$. The first disjointness relation implies, by 2.6.9, that the term is disjoint from $i(cX \& cY) = ic(X \cup Y)$; the other two imply, by 2.7.11, disjointness from $i(X \cup Y)$. Therefore the term must be included in $b(X \cup Y)$. Finally consider $bX \& bY$. This term is disjoint from $icX \& icY$ and therefore also from $ic(X \cup Y)$. It is easy to see by means of examples that $bX \& bY$ may contain points in the interior and on the boundary of $X \cup Y$. We may eliminate from $bX \& bY$ any interior points of $X \cup Y$ by intersecting it with $ci(X \cup Y) = kc(X \cup Y)$ by 2.6.7, to yield $bX \& bY \& kcX \& kcY$. The conclusion follows from our term by term analysis. Q.E.D.

**PROPERTY 2.7.13.** $b(X \& Y) = (bX \& iY) \cup (iX \& bY) \cup (bX \& bY \& k(X \& Y))$.

**PROOF:** Using 2.7.5 and 2.7.12 yields $b(X \& Y) = bc(X \& Y) = b(cX \cup cY) = (bcX \& iccY) \cup (iccX \& bcy) \cup (bcX \& bcy \& k(ccX \& ccY))$. The conclusion follows readily from 2.7.5. Q.E.D.

**PROPERTY 2.7.14.** $b(X - Y) = (bX \& icY) \cup (iX \& bcy) \cup (bX \& bcy \& k(X \& cy))$.

**PROOF:** Applying 2.7.13 to $X - Y = X \& cy$ yields $b(X - Y) = (bX \& icY) \cup (iX \& bcy) \cup (bX \& bcy \& k(X \& cy))$, and the conclusion follows from 2.7.5. Q.E.D.

*Properties 2.7.12 - 2.7.14 apply to arbitrary sets X,Y. If X and Y are closed 2.7.12 and 2.7.14 may be simplified slightly (by using the fact that $icX = cX$ for closed X), and 2.7.13 may be simplified significantly, as follows.*

**PROPERTY 2.7.15.** If X and Y are closed then $b(X \& Y) = (bX \& iY) \cup (iX \& bY) \cup (bX \& bY) = (X \& bY) \cup (bX \& Y)$.

**PROOF:** Because X and Y are closed, so is $X \& Y$ and therefore $k(X \& Y) = X \& Y = kX \& kY$. Because of 2.7.6 $bX \& bY$ is a subset of $kX \& kY$, and the conclusion follows. Q.E.D.

* *

2.8 SETS WITH NOWHERE DENSE BOUNDARY

Consider the set X of all rationals in the real interval $(0,1)$. In the usual topology of the real line we have: $iX = \ldots$
\(0, kX = [0,1],\) and \(bX = [0,1].\) This example shows that the intuitive notion that the boundary of a set is a "thin set" which "separates" the set from its complement is not always adequate. In this section we introduce a class of sets whose boundaries are suitably "thin" for the intuitive concept of boundary to be "justifiable".

**DEFINITION 2.8.1.** A subset \(X\) of \(W\) is **NOWHERE DENSE** if \(1kX = \emptyset\) [4].

*Note that nowhere dense sets have empty interior but the converse is not true in general. For example, the set of rationals has empty interior and yet it is not a nowhere dense set; the interior of its closure is the whole real line.*

**PROPERTY 2.8.2.** A subset of a nowhere dense set is also nowhere dense. The intersection of two nowhere dense sets is also nowhere dense.

**PROOF:** The second part of the property follows from the first since \(X \& Y\) is a subset of \(X\). The first part is a consequence of Properties 2.5.7 and 2.6.5. Q.E.D.

**DEFINITION 2.8.3.** A subset \(X\) of \(W\) has **NOWHERE DENSE BOUNDARY** if \(bX\) is nowhere dense, i.e., if \(ibX = \emptyset\).

**PROPERTY 2.8.4.** Closed sets have nowhere dense boundary [4].

**PROOF:** Let \(X\) be closed; \(bX\) is included in \(X = kX\). Suppose that \(ibX\) is not empty. Then \(iX \cup ibX\) is an open set strictly larger than \(iX\) (because \(ibX\) is disjoint of \(iX\)) and included in \(X\), which is a contradiction by 2.6.3. Q.E.D.

**PROPERTY 2.8.5.** Open sets have nowhere dense boundaries [5].

**PROOF:** Let \(X\) be open. Then \(bX = bcX\) by 2.7.5, and \(bcX\) is nowhere dense by 2.8.4 since \(cX\) is a closed set. Q.E.D.

**PROPERTY 2.8.6.** If \(X\) and \(Y\) are subsets of \(W\) having a nowhere dense boundary, then \(i(X \cup U Y)\) is a subset of \(k(iX \cup U Y)\).

**PROOF:** By contradiction. Let \(x\) be a point of \(i(X \cup U Y)\) but not of \(k(iX \cup U Y)\). The first condition implies that there is a neighborhood \(N'\) of \(x\) which is included in \(i(X \cup U Y)\); the second condition implies that there is a
neighborhood $N^\prime$ of $x$ which is included in $l(x \cup y) = c(x \cup y)$, and therefore $N^\prime$ does not intersect $x \cup y$.
Consider $N = N^\prime \cap N^\prime$; $N$ is also a neighborhood of $x$ and has the properties derived above for $N^\prime$ and $N^\prime$. In particular $N$ is included in $l(x \cup y)$ and hence, by Property 2.7.11, is contained in $(x \cup y) \cup (b \times 8 \times b \cup y)$. But $N$ does not intersect $x \cup y$. We reach a contradiction because $N$ cannot be included in $b \times 8 \times b \cup y$ which is a nowhere dense set by the hypothesis and 2.8.2. Q.E.D.

PROPERTY 2.8.7. The union of two nowhere dense sets is also nowhere dense [6].

PROOF: Let $X$ and $Y$ be nowhere dense, i.e., $l(X) = 0$ and $l(Y) = 0$. Apply 2.8.6 to the sets $kX$ and $kY$, which are closed and hence, by 2.8.4, have nowhere dense boundary. It follows that $l(kX \cup kY)$ is a subset of $k(l(X) \cup l(Y))$; since the latter is empty and the first equals, by 2.5.8, $l(kX \cup Y)$ we conclude that $X \cup Y$ is nowhere dense. Q.E.D.

PROPERTY 2.8.8. The subsets of $W$ having a nowhere dense boundary form a Boolean algebra with the usual operations $U$, $\cup$, and $\cap$.

PROOF: By 2.1.7 we need only to prove that the class is closed under the operations $U$, $\cup$, and $\cap$. By 2.7.5 it is clear that $cX$ has nowhere dense boundary when $X$ does. By 2.8.7 $bX \cup bY$ is nowhere dense when $X$ and $Y$ are sets with nowhere dense boundary. This implies, by 2.7.8 and 2.8.2, that $X \cup Y$ has nowhere dense boundary. Similarly, if $X$ and $Y$ have nowhere dense boundaries, by Properties 2.7.9 and 2.8.2, $X \cup Y$ has nowhere dense boundary. Q.E.D.

PROPERTY 2.8.9. If $X$ and $Y$ are subsets of $W$ having a nowhere dense boundary then $l(kX \cup kY)$ is a subset of $k(X \cup Y)$.

PROOF: By contradiction. Suppose that $x$ is a point of $l(kX \cup kY)$ but not of $k(X \cup Y)$. Then there exist neighborhoods $N^\prime$ and $N^\prime$ of $x$ such that $N^\prime$ is included in $kX \cup kY$, and $N^\prime$ does not intersect $X \cup Y$. We can choose a neighborhood $N = N \cap N^\prime$ satisfying $N \cap (kX \cup kY) = N$. Since, by 2.7.6, $kX = X \cup bX$, and similarly for $Y$, we can expand the previous equality and conclude, by using the fact that $N$ does not intersect $X \cup Y$ that $N$ is included in $bX \cup bY$. But this contradicts the hypothesis of $X$ and $Y$ having nowhere dense boundaries because of 2.8.7. Q.E.D.

*
Properties 2.8.6 and 2.8.9 need not hold for sets whose boundaries are not nowhere dense. For example, let \( X \) be the set of rationals and \( Y \) the set of irrationals on the real line. We have \( i(X \cup Y) = i(kX \cup kY) = W \), and \( k(iX \cup iY) = k(X \cup Y) = 0 \).

**PROPERTY 2.8.10.** If \( X \) and \( Y \) have nowhere dense boundaries then \( ki(X \cup Y) = k(iX \cup iY) \).

**PROOF:** From 2.6.10 and 2.5.7 it follows that \( k(iX \cup iY) \) is included in \( ki(X \cup Y) \). The reverse inclusion follows from 2.8.6, 2.5.7, and the obvious fact that \( kkZ = kZ \) for any \( Z \). Q.E.D.

**PROPERTY 2.8.11.** If \( X \) and \( Y \) have nowhere dense boundaries then \( ik(X \cup Y) = i(kX \cup kY) \).

**PROOF:** From 2.5.9 and 2.6.5 it follows that \( ik(X \cup Y) \) is included in \( i(kX \cup kY) \). The reverse inclusion follows from 2.8.9, 2.6.5, and \( iiZ = iZ \) for all \( Z \). Q.E.D.

**2.9 TOPOLOGICALSUBSPACES**

**DEFINITION 2.9.1.** Asubset \( W' \) of the topological space \((W,T)\) is a topological subspace of \((W,T)\) with the RELATIVE or induced topology \( T' \) if the open sets \( X' \) of \( T' \) are of the form \( X' = X \cup W' \), where \( X \) is an open set of \( T \). Sets in \( T' \) are called RELATIVELY OPEN or open-\( W' \) to distinguish them from the open sets of \( T \), also called open-\( W \) [3].

**PROPERTY 2.9.2.** Let \((W',T')\) be a topological subspace of \((W,T)\). A subset \( X' \) of \( W' \) is relatively closed if and only if \( X' = X \cup W' \), for some \( X \) closed-\( W \)[3].

**It is important to note that the definitions of closure, interior, and boundary depend on the particular topology being considered. For example, let \( W = E3 \), and let \( W' \) be a planar subset of \( W \); a disk \( X \) lying in the plane \( W' \) is closed both in \( W \) and \( W' \), but the disk is nowhere dense in \( W \) and is not nowhere dense in \( W' \). Furthermore, the \( W' \) boundary of \( X \), denoted \( b'X \), is a circle, while the \( W \) boundary \( bX \) is the disk itself.**

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2.10 CONTINUITY AND HOMEOMORPHISM

DEFINITION 2.10.1. A function \( f: W \rightarrow V \), where \( (W,T) \) and \( (V,S) \) are topological spaces, is CONTINUOUS if for each \( X \) open in \( V \) the inverse image of \( X \) is open in \( W \) [3].

DEFINITION 2.10.2. A function \( f: W \rightarrow V \) is a HOMEOMORPHISM if \( f \) is continuous and possesses an inverse which is also continuous. The spaces \( W \) and \( V \) are then said to be HOMEOMORPHIC [3].

PROPERTY 2.10.3. A function \( f: W \rightarrow V \) is a homeomorphism if and only if it establishes a one to one correspondence between both the points and the open sets of \( W \) and \( V \) [3].

* *

It is often helpful to think of homeomorphisms as "elastic deformations" which preserve the "nearness" of points, although not all homeomorphisms may be interpreted in such a way.

Homeomorphic spaces are also called TOPOLOGICALLY EQUIVALENT. Topology may be viewed as the branch of mathematics which studies the properties of spaces which are invariant under homeomorphisms. Such properties are often called topological properties.

* *

2.11 COMPACTNESS AND CONNECTEDNESS

DEFINITION 2.11.1. A collection of open subsets \( X_1 \) of \( W \) is called an OPEN COVER of a subset \( Y \) of \( W \) if \( Y \) is included in the union of the \( X_1 \) [3].

DEFINITION 2.11.2. A topological space \( W \) is COMPACT if for each open covering \( (X_i) \) of \( W \) there exists a finite number of \( X_i \) which also constitute an open cover of \( W \) [3].

PROPERTY 2.11.3. (Heine-Borel) A subset of Euclidean \( n \)-space is compact if and only if it is closed and bounded [3].

This property does not hold for arbitrary metric spaces.

* *

DEFINITION 2.11.4. A topological space \( W \) is CONNECTED if it is not the union of two disjoint non-empty open sets [4].
PROPERTY 2.11.5. Compactness and connectedness are topological properties, i.e. they are preserved under homeomorphisms [3].

* * *


3. REGULAR SETS

This section is devoted to the study of closed regular sets (defined below). Because we are not interested in the so-called open regular sets we shall refer to the closed regular sets simply as regular sets. Throughout the section we assume that all sets are subsets of some topological space (W, T).

3.1 REGULARITY

DEFINITION 3.1.1. The regularization of a subset X of W, denoted rx, is the set rx = kiX.

DEFINITION 3.1.2. A set X is REGULAR if X = rx, i.e. if X = kiX [6].

* * *

It is obvious from the definitions that regular sets are closed, have nowhere dense boundaries, and that a regular set whose interior is empty must also be empty.

* *

PROPERTY 3.1.3. For any set X, rx is regular, i.e. rx = rrX [6].

PROOF: Clearly irX is contained in rx, and therefore, by 2.5.7, kirX = rrX is a subset of krX = rx. To prove the reverse inclusion note that in is an open subset of kiX and hence of kiX by 2.6.3. Therefore, by 2.5.7 kiX is a subset of kiX. Q.E.D.

* * *

3.2 REGULARIZED SET OPERATORS

It is easy to see that the class of regular sets is not closed under ∪, − and c. For example, let X = [0, 1] and Y = [1, 2]; then X ∪ Y = (1), X − Y = [0, 1), cX and cY are open, and none of these sets is regular. The class is closed (obviously) under the operators defined below.

* *

DEFINITION 3.2.1. The regularized union, intersection, difference and complement (denoted by the starred operators) are defined per
\[ X \cup^* Y = r(X \cup Y), \\
X \cap^* Y = r(X \cap Y), \\
X -^* Y = r(X - Y), \\
c^*X = rcX. \]

**PROPERTY 3.2.2.** For any subsets \( X \) and \( Y \), \( 1(X \cap^* Y) = 1(rX \cap^* rY) \).

**PROOF:** Let \( P = iX \) and \( Q = iY \). Since \( P \) and \( Q \) are open sets their boundaries are nowhere dense, and therefore we can apply 2.8.11: \( 1k(P \cap Q) = 1(kP \cap kQ) \). Using 2.6.9 and replacing \( P \) and \( Q \) by their values yields \( 1ki(X \cap Y) = 1(kiX \cap kiY) \), which implies trivially the result. Q.E.D.

**PROPERTY 3.2.3.** For any \( X \) and \( Y \), \( X \cap^* Y = rX \cap^* rY \), i.e., \( r(X \cap Y) = rX \cap^* rY \).

**PROOF:** Applying closure to both sides of equality 3.2.2 yields \( r(X \cap^* Y) = r(X \cap rY) \). This implies the conclusion because \( X \cap^* Y \) is regular and, by 3.1.3, \( r(X \cap^* Y) = X \cap^* Y \). Q.E.D.

**PROPERTY 3.2.4.** If \( X \) and \( Y \) have nowhere dense boundaries then \( X \cup^* Y = rX \cup rY \), i.e., \( r(X \cup Y) = rX \cup rY \).

**PROOF:** By 2.8.10 and 2.5.8. Q.E.D.

**PROPERTY 3.2.5.** If \( X \) and \( Y \) are regular then \( i(X \cap^* Y) = iX \cap iY \).

**PROOF:** By 3.2.2, 2.6.9, and 3.1.2. Q.E.D.

**PROPERTY 3.2.6.** If \( X \) and \( Y \) are regular then \( X \cup^* Y = X \cup Y \).

**PROOF:** By 3.2.4 and 3.1.2. Q.E.D.

*Property 3.2.6 implies that the (usual) union of regular sets is also regular. Note that there is an asymmetry between Properties 3.2.3 and 3.2.4; however, we cannot remove the restriction of nowhere dense boundaries from 3.2.4. For example, if \( X \) is the set of rationals and \( Y \) the set of irrationals \( rX \cup rY = \emptyset \) but \( r(X \cup Y) \) is the whole real line.*

*PROPERTY 3.2.7.** If \( X \) is closed (and hence also if \( X \) is regular) then \( c^*X = kcX \).
PROOF: \( cX = kicX \), and by 2.6.8 \( cX = kckX \). The result follows because \( X \) is closed and hence \( X = kX \). Q.E.D.

PROPERTY 3.2.8. The regular sets are a Boolean algebra with operators \( U^* \), \( S^* \), and \( c^* \) [6].

PROOF: We have to show that properties 2.1.1 through 2.1.4 hold for regular \( X \), \( Y \), \( Z \), with the operators replaced by \( U^* \), \( S^* \), and \( c^* \). Firstly note that \( \emptyset \) and \( W \) are clearly regular. Commutativity follows trivially from the definition of regularized operators. Property 2.1.3 is obviously satisfied. Also, by 3.2.6 and 3.2.7 \( X U^* c^X = X U kckX = W \). To complete the proof that 2.1.4 is satisfied note that \( X S^* c^X = r(X S kckX) \) by 3.2.7. Using 2.7.3 and the fact that \( X \) is closed and hence has nowhere dense boundary yields \( X S^* c^X = kibX = \emptyset \) (because nowhere dense sets have empty interior). Finally, let us consider the distributive properties 2.1.2. We have \( X U^* (Y S Z) = rX U r(X S Z) \) by 3.2.6 and 3.2.1. Since both \( X \) and \( Y S Z \) are closed and therefore have nowhere dense boundary we can use 3.2.4 and conclude that \( X U^* (Y S Z) = r[X U (Y S Z)] = r[(X U Y) S (X U Z)] = (X U^* Y) S^* (X U^* Z) \). Now consider \( X S^* (Y U^* Z) = r[(X S Y) U (X S Z)] = r(X S Y) U r(X S Z) = (X S^* Y) U^* (X S^* Z) \), where we have used the fact that \( X S Y \) and \( X S Z \) have nowhere dense boundaries. This concludes the proof. Q.E.D.

PROPERTY 3.2.9. For any \( X \) and \( Y \), \( X -\# Y = X * cY \).

PROOF: From definition 3.2.1, and the properties of the usual set difference, it follows that \( X -\# Y = r(X - Y) = r(X S cY) = X S^* cY \). Q.E.D.

PROPERTY 3.2.10. If \( X \) and \( Y \) are regular \( X -\# Y = X S^* cY \).

PROOF: From 3.2.9 and 3.2.3 \( X -\# Y = X S^* cY = rX S^* rcY \). The conclusion follows since \( X = rX \) is regular and \( rcY = cY \) by definition 3.2.1. Q.E.D.

Property 3.2.10 establishes that the regularized difference of regular sets \( X \), \( Y \) (defined per 3.2.1) is identical to the difference \( X S^* cY \) implied by the Boolean algebra of 3.2.8. For reasons explained in [2], the Boolean algebra is less appropriate for some applications than a ring of sets with operators \( U^* \), \( S^* \), and \( -\# \). The algebraic properties of the ring follow from those of the Boolean algebra or may be established directly.
3.3 RECURSION FORMULAE FOR THE INTERIOR, BOUNDARY AND COMPLEMENT

For reasons discussed at length in [2], [7] and [8] it is important to relate the interiors, boundaries, and complements of sets to the interiors, boundaries, and complements of their regularized "compositions". We already derived one of such "recursion formulae" for the interior of regularized intersections in 3.2.5. Others are discussed below.

*

PROPERTY 3.3.1. If $X$ and $Y$ are regular then $c(X \cup^* Y) = cX \& cY$.

PROOF: Because of 3.2.6, this is merely De Morgan's law.
Q.E.D.

PROPERTY 3.3.2. If $X$ is regular then $ic^*X = cX$.

PROOF: By 2.6.6 and 3.2.7 $ic^*X = cckcX = ck(cckX) = cklX = crX$, and this equals $cX$ since $X$ is regular, i.e. $X = rX$. Q.E.D.

PROPERTY 3.3.3. If $X$ is regular then $bc^*X = bX$.

PROOF: By 2.7.3 and 3.2.7 $bc^*X = bkcX = kkcX \& kckcX = kcX \& klcX$; the last transformation used also 2.6.6. The conclusion follows from $X = klcX$ and 2.7.3. Q.E.D.

PROPERTY 3.3.4. If $X$ is regular then $cc^*X = iX$.

PROOF: By 3.2.7 $cc^*X = ckcX$, which equals $iX$ by 2.6.6.
Q.E.D.

PROPERTY 3.3.5. If $X$ and $Y$ are regular then $i(X \ast^* Y) = iX \& iY$.

PROOF: Using 3.2.10 we can write $X \ast Y = X \& cY$. It follows from 3.2.5 that $i(X \& cY) = iX \& ic^*Y$, which, by 3.3.2 implies the conclusion. Q.E.D.

PROPERTY 3.3.6. If $X$ and $Y$ are regular then $b(X \cup^* Y) = (bX \& cY) U (cX \& bY) U [bX \& bY \& k(cX \& cY)]$.

PROOF: Using 3.2.6 and the fact that $X, Y$ are closed the conclusion follows immediately from 2.7.12. Q.E.D.

PROPERTY 3.3.7. If $X$ and $Y$ are regular then $b(X \& Y) = (bX \& iY) U (bY \& iX) U [bX \& bY \& k(iX \& iY)]$. 
PROOF: By 3.3.3 \( b(X \& Y) = bc^*(X \& Y) \), which equals, by De Morgan's law for regular sets, \( b(c^*X \cup c^*Y) \). Now we apply 3.3.6 and obtain \( (bc^*X \& cc^*Y) \cup (cc^*X \& bc^*Y) \cup [bc^*X \& bc^*Y \& k(cc^*X \& cc^*Y)] \). By using 3.3.4 and 3.3.3, the expression above may be simplified to yield the conclusion. Q.E.D.

PROPERTY 3.3.8. If \( X \) and \( Y \) are regular then \( b(X ^{-} Y) = (bX \& cY) \cup (1X \& bY) \cup [bX \& bY \& k(1X \& cY)] \).

PROOF: We can write \( X ^{-} Y = X \& cY \) and apply 3.3.7. This yields \( b(X ^{-} Y) = (bX \& 1cY) \cup (1X \& bcY) \cup [bX \& bcY \& k(1X \& 1cY)] \), which may be simplified by 3.3.2 and 3.3.3 to the expression in the conclusion. Q.E.D.

PROPERTY 3.3.9. If \( X \) and \( Y \) are regular, then \( 1(X U^* Y) = 1X U 1Y U [bX \& bY \& ck(cX \& cY)] \).

PROOF: By 3.2.6 \( 1(X U^* Y) = 1(X U Y) \). From 2.6.10 and 2.7.11 it follows that \( 1(X U Y) = 1X U 1Y U [bX \& bY \& 1(X U Y)] \). Using 2.6.6 and De Morgan's law the right-hand side of this expression may be written as \( 1X U 1Y U [bX \& bY \& ck(cX \& cY)] \). Q.E.D.

PROPERTY 3.3.10. If \( X \) and \( Y \) are regular then \( c(X \& Y) = cX U cY U [bX \& bY \& ck(1X \& 1Y)] \).

PROOF: By 3.3.2 \( c(X \& Y) = ic^*(X \& Y) \) which, by De Morgan's law for regular sets, equals \( ic^*(X \& Y) \). Applying 3.3.9 yields \( 1(c^*X U^* c^*Y) = ic^*X U ic^*Y U [bc^*X \& bc^*Y \& ck(cc^*X \& cc^*Y)] \). The result now follows from 3.3.2, 3.3.3, and 3.3.4. Q.E.D.

PROPERTY 3.3.11. If \( X \) and \( Y \) are regular, then \( c(X ^{-} Y) = cX U cY U [bX \& bY \& ck(1X \& 1Y)] \).

PROOF: \( c(X ^{-} Y) = c(X \& cY) = cX U ccY U [bX \& bcY \& ck(1X \& 1cY)] \) by Property 3.3.10. The result follows from Properties 3.3.4, 3.3.3, and 3.3.2. Q.E.D.

* 

It is interesting to compare the general boundary formulae (Properties 2.7.12 - 2.7.15) with the boundary formulae for regularized operators and regular sets (Properties 3.3.6 - 3.3.8). For example, if \( X, Y \) are regular (hence closed) \( b(X \& Y) \) is given by Property 2.7.15 while \( b(X ^{-} Y) \) is given by Property 3.3.7. Observe that \( b(X \& Y) \) contains all of \( bX \& bY \), while \( b(X ^{-} Y) \) contains only those points of \( bX \& bY \) that intersect \( k(1X \& 1Y) \).

* * *
4. MEMBERSHIP CLASSIFICATION

The membership classification function, defined below, is a generalization and formalization of "clipping". Practical applications of classification are discussed in [1], [2], [7], and [8]. In this section we shall define the function and establish some of its formal properties.

Throughout the section we let \((W', T')\) be a topological subspace of \((W, T)\). We shall use primed symbols (e.g. \(k', i', r'\), \(s'\)) to denote operations in the \((W', T')\) space. As indicated in section 2.9, it is important to bear in mind the distinction between topological concepts in the two spaces.

*  

4.1 THE MEMBERSHIP CLASSIFICATION FUNCTION

DEFINITION 4.1.1. If \(X\) is regular-\(W'\) and \(S\) is regular-\(W\), then the MEMBERSHIP CLASSIFICATION FUNCTION, \(M(X, S)\), is defined as follows:

\[ M(X, S) = (X_{in S}, X_{on S}, X_{out S}) \]

where

\[ X_{in S} = X \ast' \ W_{in} = r' (X \wedge W_{in}) \]
\[ X_{on S} = X \ast' \ W_{on} = r' (X \wedge W_{on}) \]
\[ X_{out S} = X \ast' \ W_{out} = r' (X \wedge W_{out}) \]

We shall refer to the 3-tuple of classification results, \((X_{in S}, X_{on S}, X_{out S})\), as the classification of \(X\) with respect to \(S\). \(X\) is called the CANDIDATE set and \(S\) is called the REFERENCE set. Observe that each of the classification results is a regular-\(W'\) subset of the candidate.

*

PROPERTY 4.1.2. If \(X\) is regular-\(W'\), \(S\) is regular-\(W\) and \(M(X, S) = (X_{in S}, X_{on S}, X_{out S})\) then:

a) \(X = X_{in S} \cup X_{on S} \cup X_{out S}\)

b) \(X_{in S} \ast' X_{on S} = X_{on S} \ast' X_{out S} = X_{in S} \ast' X_{out S} = 0\).

PROOF: For the first part, \(X_{in S} \cup X_{on S} \cup X_{out S} = r' (X \wedge W_{in}) \cup r' (X \wedge W_{on}) \cup r' (X \wedge W_{out})\). Firstly we shall show that \(X \wedge W_{in}, X \wedge W_{on}\) and \(X \wedge W_{out}\) have nowhere dense boundaries in \(W'\). Note that \(X \wedge W_{in} = X \wedge (W_{in} \wedge W')\), \(X \wedge W_{on} = X \wedge (W_{on} \wedge W')\), and \(X \wedge W_{out} = X \wedge (W_{out} \wedge W')\). By 2.9.1 and 2.9.2 \(W_{in}, W_{on}\) and \(W_{out}\) are open-\(W'\) while \(W_{in}, W_{on}\) and \(W_{out}\) are open-\(W'\). This implies, by 2.8.4 and 2.8.5, that these sets have nowhere dense boundaries in \(W'\). Since \(X\)
also has nowhere dense boundary in $W'$, by 2.8.8, $X \& (IS \& W')$, $X \& (bS \& W')$, and $X \& (cS \& W')$ have nowhere dense boundaries in $W'$. Now we may apply Property 3.2.4 to obtain $XinS \cup XonS \cup XoutS = r'(X \& (IS \& bS \& cS)) = r'(X \& W)$ by Property 2.7.7 since $S$ is closed-$W$ hence $cS = IC\$/S$. Since $X$ is contained in $W'$ which is contained in $W$, $r'(X \& W) = r'(X) = X$ because $X$ is regular-$W'$. For the second part, $XinS \& XonS = r'(X \& IS) \& XonS = r'((X \& IS) \& bS)$; by Property 3.2.3, the right-hand side equals $(X \& IS) \& XonS = r'((X \& IS) \& bS)$. Since $is$ and $bS$ are mutually disjoint, we have $XinS \& XonS = r'(\emptyset) = \emptyset$. The other two "cross products" are null by similar reasoning. Q.E.D.

This establishes that the classification results form what we might think of as a regular-$W'$ decomposition of the candidate.

* *

PROPERTY 4.1.3. If $X$ is regular-$W'$, $S$ is regular-$W$ and $M\{X,S\}$ = $(XinS, XonS, XoutS)$ then:

a) $XinS$ is a subset of $r'(IS \& W')$.

b) $XonS$ is a subset of $r'(bS \& W')$.

c) $XoutS$ is a subset of $r'(cS \& W')$.

PROOF: We shall prove the first part; the two other assertions may be proved similarly. $XinS = r'(X \& IS) = r'(X \& IS \& W')$ since $X$ is contained in $W'$. Since $IS \& W'$ is a subset of $W'$, we may apply Property 3.2.3 to obtain $XinS = X \& XonS = r'(IS \& W')$. Using Property 3.2.5, it follows that $i'XinS = i'X \& i'XonS = r'(IS \& W')$ is contained in $i'X \& W'$. Hence, by Property 2.5.7 $k'1'i'XinS$ is contained in $k'1'i'X \& W'$. The result follows from the fact that $XinS$ and $r'(IS \& W')$ are both regular-$W'$ and Property 3.1.3. Q.E.D.

This property establishes that the classification results lie respectively in the regularization of the intersection of the interior, boundary and complement of the reference set with the "small world" $W'$. Stronger containment relations need not be satisfied in general. For example, it is not generally the case that $XinS$ is contained in $IS$ or that $i'XinS$ is contained in $IS$ as the following counter example demonstrates. Let $W$ be the real line with the usual topology, $W'$ be $[0,2]$, $S$ be $[0,2]$ and $X$ be $[0,2]$. In this case $XinS = [0,2] = i'XinS$ since $W'$ is an open set. However, $IS = (0,2)$.

* *
4.2 RECURSION FORMULAE FOR THE MEMBERSHIP CLASSIFICATION FUNCTION

For reasons discussed in [1], [2], [7], and [8] it is important to relate the classification of \( X \) with respect to a pair of reference sets, \( A \) and \( B \), to the classification of \( X \) with respect to a set \( S = A \circ B \) or \( A \cup B \) or \( A \setminus B \). The next three properties establish the desired formulae.

PROPERTY 4.2.1. If \( X \) is regular-\( W \), \( A \) and \( B \) are regular-\( W \), \( M(X,A) = (XinA, XinB, XoutA), M(X,B) = (XinB, XinB, XoutB) \) and \( S = A \circ B \), then:

a) \( XinS = XinA \circ XinB \).

b) \( XonS = (XonA \circ XinB) U (XinA \circ XonB) U r'(p in (XonA \circ XinB) such that every neighborhood of \( p \) in \( W \) intersects \( iA \circ iB \)).

c) \( XoutS = XoutA U (XoutA U XoutA U r'(p in (XonA \circ XinB) such that there exists a neighborhood of \( p \) in \( W \) disjoint from \( iA \circ iB \)).

PROOF: Part a) follows from Properties 3.2.5 and 3.2.3 since \( XinS = r'(X \circ iS) = r'(X \circ iA \circ iB) = r'[X \circ iA \circ iB] = r'[r'(X \circ iA) \circ r'(X \circ iB)] = XinA \circ XinB \).

For part b) we may use Property 3.3.7 to obtain \( XinS = r'(X \circ bS) = r'[(X \circ bA \circ iB) U (X \circ iA \circ bB) U (X \circ bA \circ bB \circ k(iA \circ iB))] \). Since \( X \circ bA \circ iB = X \circ (bA \circ W) \circ (iB \circ W) \), it is the intersection of two closed-\( W \) sets with an open-\( W \) set (by 2.9.1 and 2.9.2) and therefore it has nowhere dense boundary in \( W \) (by 2.6.4, 2.8.5 and 2.8.6). Similar arguments hold for the other terms in the expansion of \( XinS \), and we may apply Property 3.2.4 to obtain \( XinS = r'(X \circ bA \circ iB) U r'(X \circ iA \circ bB) U r'(X \circ bA \circ bB \circ k(iA \circ iB)) \). From 3.2.3 we have \( r'(X \circ bA \circ iB) = XonA \circ XinB \).

Similarly, \( r'(X \circ iA \circ bB) = XinA \circ XonB \circ XinB \). From 3.2.3 it follows that \( r'[X \circ bA \circ bB \circ k(iA \circ iB)] = r'(X \circ bA) \circ XinB \circ XinB \circ XinB \circ XinB \). Therefore, \( XinA \circ XinB \), another application of 3.2.3 yields \( r'[(XonA \circ XinB) \circ (X \circ k(iA \circ iB))] \) which equals \( r'[XonA \circ XinB \circ XinB \circ XinB \circ XinB] \) because \( XinA \circ XinB \) is a subset of \( X \). Finally, applying the definition of closure, we obtain \( r'(X \circ bA \circ bB \circ k(iA \circ iB)) = r'(p in (XonA \circ XinB) such that every neighborhood of \( p \) in \( W \) intersects \( iA \circ iB \)), and the result follows. Part c) follows from Property 3.3.10 in a similar fashion.
Briefly, \( \text{Xout}S = r'(X \circ cS) = r'(X \circ cA) U (X \circ cB) U [X \circ bA \circ bB \circ ck(iA \circ iB)] = \text{XoutA} U \text{XoutB} U r'(p \text{ in } \text{XonA} \circ XonB \text{ such that there exists a neighborhood of } p \text{ in } W \text{ that is disjoint from } iA \circ iB). \) Q.E.D.

PROPERTY 4.2.2. If \( X \) is regular-\( W' \), \( A \) and \( B \) are regular-\( W \), \( M[X,A] = (\text{XinA}, \text{XonA}, \text{XoutA}), M[X,B] = (\text{XinB}, \text{XonB}, \text{XoutB}) \) and \( S = A U* B \), then:

a) \( \text{Xin}S = \text{Xin}A U \text{Xin}B U r'(p \text{ in } (\text{XonA} \circ XonB) \text{ such that there exists a neighborhood of } p \text{ in } W \text{ that is disjoint from } cA \circ cB). \)

b) \( \text{Xon}S = (\text{XonA} \circ XonB) U (\text{XoutA} \circ XonB) U r'(p \text{ in } (\text{XonA} \circ XonB) \text{ such that every neighborhood of } p \text{ in } W \text{ intersects } cA \circ cB). \)

c) \( \text{Xout}S = \text{XoutA} \circ XonB. \)

PROOF: Parts a), b), c) follow from properties 3.3.9, 3.3.6 and 3.3.1, respectively. The proof is similar to that of 4.2.1. Q.E.D.

PROPERTY 4.2.3. If \( X \) is regular-\( W' \), \( A \) and \( B \) are regular-\( W \), \( M[X,A] = (\text{XinA}, \text{XonA}, \text{XoutA}), M[X,B] = (\text{XinB}, \text{XonB}, \text{XoutB}) \) and \( S = A -* B \), then:

a) \( \text{Xin}S = (\text{XonA} \circ XonB) U (\text{XinA} \circ XonB) U r'(p \text{ in } (\text{XonA} \circ XonB) \text{ such that every neighborhood of } p \text{ in } W \text{ intersects } iA \circ cB). \)

b) \( \text{Xon}S = \text{XoutA} U \text{XinB} U r'(p \text{ in } (\text{XonA} \circ XonB) \text{ such that there exists a neighborhood of } p \text{ in } W \text{ that is disjoint from } iA \circ cB). \)

c) \( \text{Xout}S = \text{XoutA} U \text{XinB} U r'(p \text{ in } (\text{XonA} \circ XonB) \text{ such that there exists a neighborhood of } p \text{ in } W \text{ that is disjoint from } iA \circ cB). \)

PROOF: Parts a), b), c) follow from Properties 3.3.5, 3.3.8, and 3.3.11, respectively. The proof is similar to that of Property 4.2.1. Q.E.D.

\* \*
4.3 USE OF BALL NEIGHBORHOODS FOR CLASSIFICATION IN METRIC SPACES

Properties 4.2.1 - 4.2.3 provide recursion formulae for determining the classification of a candidate X with respect to a reference set $S = A \langle \text{op} \rangle B$, where $\langle \text{op} \rangle$ is a regularized operator in W, given the classification of X with respect to A and B plus certain information about neighborhoods of points in $\text{XonA} \ast \ast \text{XonB}$. These properties were derived for general topological spaces. In certain metric spaces it is more convenient to deal with regular balls than with general neighborhoods. Firstly we shall see that properties 4.2.1 to 4.2.3 may be rephrased for metric spaces in terms of open balls by using the following property.

PROPERTY 4.3.1. Let $(W, d)$ be a metric space and let $(W, T)$ be the natural topological space associated with $(W, d)$. Let $X$ be a subset of $W$ and let $p$ be some point in $W$. Then:

a) Every neighborhood of $p$ in $(W, T)$ intersects $X$ if and only if every open ball about $p$ in $(W, d)$ intersects $X$.

b) There exists a neighborhood of $p$ in $(W, T)$ disjoint from $X$ if and only if there exists an open ball about $p$ in $(W, d)$ disjoint from $X$.

PROOF: Firstly note that b) is the contrapositive of a) and therefore follows logically from a). To prove a) suppose that every neighborhood of $p$ in $(W, T)$ intersects $X$. Then every open ball about $p$ in $(W, d)$ intersects $X$ since every open ball about $p$ is a neighborhood of $p$. Conversely, suppose that every open ball about $p$ intersects $X$. Let $N(p)$ be a neighborhood of $p$ in $(W, T)$. $N(p)$ contains an open set containing $p$. Since the open sets of $(W, T)$ are precisely the "metrically" open sets of Definition 2.2.3, it follows that $N(p)$ contains an open ball about $p$. Hence $N(p)$ intersects $X$. Q.E.D.

As an example of the use of Property 4.3.1, in a metric space we may replace the condition in part b) of 4.2.1 "every neighborhood of $p$ in $W$ intersects $iA \ast iB$" by the condition "for every positive $R$, $B(p; R) \ast iA \ast iB$ is non-empty". In many metric spaces the neighborhood conditions of Properties 4.2.1 - 4.2.3 may be expressed in yet another way using regularity concepts.

PROPERTY 4.3.2. Let $(W, d)$ be a metric space with the property that $B(p; R) = iKB(p; R)$ for all $p$ in $W$ and all positive $R$. If $A$ and $B$ are regular sets in the topological space associated with $(W, d)$ then, for any $p$ in $W$ and any positive $R$, $B(p; R) \ast iA \ast iB$ is non-empty if and only if $KB(p; R) \ast A \ast B$ is non-empty.
PROOF: First, suppose that $B(p;R) \& iA \& iB$ is nonempty. Observe that $kB(p;R)$ is regular since $ikkB(p;R) = kB(p;R)$ by the hypothesis of the property. Hence, using Property 3.2.5 we obtain $ikkB(p;R) \& A \& B = ikkB(p;R) = iA \& iB = B(p;R)$ which is non-empty. Hence $kB(p;R) \& A \& B$ is non-empty. Conversely, suppose that $kB(p;R) \& A \& B$ is non-empty. Since this is a regular set, its interior must be non-empty. The result follows from Property 3.2.5. Q.E.D.

Observe that Property 4.3.2 is valid only in a metric space with the property that every open ball is equal to the interior of its closure. An example of a metric space that does not have this property is the set of reals in the interval $[0,2]$ with the usual distance function. The open ball about $p=1$ of radius $R=1$ is $(0,2)$. In this space $ik(0,2) = i[0,2] = [0,2]$ (the universe is an open set!) which is not equal to $(0,2)$. Essentially, the hypothesis of Property 4.3.2 is valid in any metric space with the property that for any distinct points $p$ and $q$, and any positive $R$, $B(p;R)$ contains some point $s$ such that $d(q,s)$ is greater than $d(p,q)$. In our example above, there is no point in an open ball about $p=0$ whose distance to the point $q=1$ (or any other $q$ in $(0,2)$) for that matter is greater than the distance between $p$ and $q$. Property 4.3.2 is useful because many metric spaces of interest satisfy the hypothesis. Examples include $E^3$ and subspaces of $E^3$ (with the relative topology) such as an unbounded plane or a spherical surface. Note carefully, however, that subsets of $E^3$ such as a bounded portion of a plane or of a sphere (considered as topological spaces with the relative topology) do not satisfy the hypothesis.


Property 4.3.2, allows us to "rewrite" Properties 4.2.1 - 4.2.3 in terms of regular ball neighborhoods. In Properties 4.3.3 - 4.3.5 below we assume that $(W,d)$ is a metric space with the property that every open ball equals the interior of its closure, $(W,T)$ is the natural topological space associated with $(W,d)$ and $(W',T')$ is a topological subspace of $(W,T)$. We also assume that $X$ is regular-$W'$, $A$ and $B$ are regular-$W$, $MX,A1 = (XinA, XinA, XoutA)$ and $MX,B1 = (XinB, XinB, XoutB)$

PROPERTY 4.3.3. Let $S = A \& B$. Then

a) $XinS = XinA \& XinB$.

b) $XonS = (XonA \& XinB) \cup (XinA \& XonB) \cup r'(p \in (XonA \& XinB) \text{ such that } kB(p;R) \& A \& B \text{ is non-empty for every positive } R)$.
c) \( X_{outS} = X_{outA} U \\
     X_{outB} U \\
     r'(p \text{ in } (X_{onA} 8* X_{onB}) \text{ such that } kB(p;R) 8* \\
     A 8* B \text{ is empty for some positive } R). \\
\)

PROOF: Part b) follows from 4.2.1 and 4.3.1 since, by 
4.3.2, \( kB(p;R) 8* A 8* B \text{ is non-empty for every } R \neq \emptyset \) if 
and only if \( B(p,R) \neq \emptyset \); \( A \neq \emptyset \); \( B \neq \emptyset \). 
Part c) is proved similarly. Q.E.D.

PROPERTY 4.3.4. Let \( S = A U* B \). Then

a) \( X_{inS} = X_{inA} U \\
     X_{inB} U \\
     r'(p \text{ in } (X_{onA} 8* X_{onB}) \text{ such that } kB(p;R) 8* \\
     c*A 8* c*B \text{ is empty for some positive } R). \\
\)

b) \( X_{onS} = (X_{onA} 8* X_{onB}) U \\
     (X_{outA} 8* X_{outB}) U \\
     r'(p \text{ in } (X_{onA} 8* X_{onB}) \text{ such that } kB(p;R) 8* \\
     c*A 8* c*B \text{ is non-empty for all positive } R). \\
\)

c) \( X_{outS} = X_{outA} 8* X_{outB}. \\
\)

PROOF: Because of 4.3.2 and 3.3.2 \( kB(p;R) 8* c*A 8* c*B \text{ is empty if and only if } \\
B(p,R) \neq \emptyset \text{ for all positive } R) \).

PROPERTY 4.3.5. Let \( S = A \text{ -- } B \). Then

a) \( X_{inS} = X_{inA} 8* X_{outB}. \\
\)

b) \( X_{onS} = (X_{onA} 8* X_{outB}) U \\
     (X_{inA} 8* X_{onB}) U \\
     r'(p \text{ in } (X_{onA} 8* X_{onB}) \text{ such that } kB(p;R) 8* \\
     A 8* c*B \text{ is non-empty for all positive } R). \\
\)

c) \( X_{outS} = X_{outA} U \\
     X_{inB} U \\
     r'(p \text{ in } (X_{onA} 8* X_{onB}) \text{ such that } kB(p;R) 8* \\
     A 8* c*B \text{ is empty for some positive } R). \\
\)

PROOF: The proof is similar to those of Properties 4.3.3 
and 4.3.4. It is based on the fact that \( kB(p;R) 8* A 8* \\
c*B \text{ is empty if and only if } B(p,R) \neq \emptyset \); \( A \neq \emptyset \); 
Q.E.D.

*
Properties 4.3.3 - 4.3.5 show that the point set XonA $\ast^\prime$ XonB is usually not wholly in either of XinS, XonS, or XoutS. Segmentation of XonA $\ast^\prime$ XonB is guided by the behavior of expressions such as $kB(p;R) \ast^\prime A \ast c*B = [kB(p;R) \ast A] \ast [kB(p;R) \ast c*B]$. The latter formulation is quite useful computationally because it allows segmentation decisions for M[X,S] to be made through computations involving regularized operations on terms of the form $N(p,A;R) = kB(p;R) \ast A$. The term $N(p,A;R)$ is called a "neighborhood with respect to A" because it is a neighborhood of p in the topological subspace A (with the relative topology). Therefore XonA $\ast^\prime$ XonB may be segmented by studying the results of "combining" neighborhoods with respect to the "component" sets A and B [2,7,8].

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REFERENCES


