EXPECTED STOCK RETURNS
AND VOLATILITY

by
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Comments Welcome

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Abstract

This paper examines the relation between stock returns and stock market volatility. We find evidence that the expected market risk premium (the expected return on a stock portfolio minus the Treasury bill yield) is positively related to the predictable volatility of stock returns. There is also evidence that unexpected stock market returns are negatively related to the unexpected change in the volatility of stock returns. This negative relation provides indirect evidence of a positive relation between expected risk premiums and volatility.
1. Introduction

Many studies document cross-sectional relations between risk and expected returns on common stocks. These studies generally measure a stock's risk as the covariance between its return and one or more variables. For example, the expected return on a stock is found to be related to covariances between its return and (i) the return on a market portfolio [Black, Jensen, and Scholes (1972), Fama and MacBeth (1973)], (ii) factors extracted from a multivariate time series of returns [Roll and Ross (1980)], (iii) macroeconomic variables, such as industrial production and changes in interest rates [Chen, Roll, and Ross (1986)], and (iv) aggregate consumption [Breeden, Gibbons, and Litzenberger (1986)].

We examine the intertemporal relation between risk and expected returns. In particular, we ask whether the expected market risk premium, defined as the expected return on a stock market portfolio minus the risk-free interest rate, is positively related to the volatility of the stock market.

Some argue that the relation between expected returns and volatility is strong. For example, Pindyck (1984) attributes much of the decline in stock prices during the 1970's to increases in risk premiums arising from increases in volatility. On the other hand, Poterba and Summers (1986) argue that the time series properties of volatility make this scenario unlikely. However, neither study provides a direct test of the relation between expected risk premiums and volatility.

We investigate relations of the form,

$$ E(R_{mt} - R_{ft} | \hat{\sigma}_{mt}^2) = \alpha + \beta \hat{\sigma}_{mt}^2, \quad p = 1, 2 $$

where $R_{mt}$ is the return on a stock market portfolio, $R_{ft}$ is the risk-free interest rate, $\hat{\sigma}_{mt}^2$ is an ex ante measure of the portfolio's standard deviation, and $\hat{\sigma}_{mt}^2$ is an ex ante measure of the variance. If $\beta = 0$ in (1), the expected risk premium is unrelated to the ex ante volatility. If $\alpha = 0$
and $\beta > 0$, the expected risk premium is proportional to the standard deviation ($p = 1$) or variance ($p = 2$) of stock market returns.

Merton (1980) estimates the relation between the market risk premium and volatility with a model similar to (1). However, given the exploratory nature of his study, he does not test whether $\beta$ equals zero. Merton also uses contemporaneous, rather than ex ante, measures of volatility, so his measures include both ex ante volatility and the unexpected change in volatility. We argue below that a positive relation between the expected risk premium and ex ante volatility will induce a negative relation between the unexpected premium and the unexpected change in volatility. Therefore, combining the two components of volatility obscures the ex ante relation.

This study uses two different statistical approaches to investigate the relation between expected stock returns and volatility. In the first, we use daily returns to compute estimates of monthly volatility. We decompose these estimates into predictable and unpredictable components using univariate autoregressive-integrated-moving-average (ARIMA) models. Regressions of the monthly risk premium on the predictable component provide little evidence of a positive relation between ex ante volatility and expected risk premiums. However, there is a strong negative relation between unexpected risk premiums and the unpredictable component of volatility. We interpret this as indirect evidence of a positive ex ante relation.

We also use daily returns to estimate ex ante measures of volatility with a generalized autoregressive conditional heteroskedasticity (GARCH) model [Engle (1982), Bollerslev (1986)]. The GARCH-in-mean approach of Engle, Lilien, and Robins (1984) is used to estimate the ex ante relation between risk premiums and volatility. These results support our interpretation of the ARIMA results. The Garch-in-mean approach indicates a reliable positive relation.
between expected risk premiums and volatility.

2. Time Series Properties of the Data

2.1 Standard Deviations of Stock Market Returns

We use daily values of the Standard and Poor's (S&P) composite portfolio to estimate the monthly standard deviation of stock market returns from January 1928 through December 1984. This estimator has three advantages over the rolling 12-month standard deviation used by Officer (1973) and by Merton (1980) over his full 1926-78 sample period. (Merton uses daily returns to estimate monthly standard deviations for 1962-1978.) First, by sampling the return process more frequently, we increase the accuracy of the standard deviation estimate for any particular interval. Second, the volatility of stock returns is not constant. We obtain a more precise estimate of the standard deviation for any month by using only returns within that month. Finally, our monthly standard deviation estimates use non-overlapping samples of returns, whereas adjacent rolling 12-month estimators share 11 returns.

Nonsynchronous trading of securities causes daily portfolio returns to be autocorrelated, particularly at lag 1 [see Fisher (1966) and Scholes and Williams (1977)]. Because of this autocorrelation, we estimate the variance of the monthly return to the S&P portfolio as the sum of the squared daily returns plus twice the sum of the products of adjacent returns,

\[ \sigma^2_{mt} = \sum_{i=1}^{N_t} r_{it}^2 + 2 \sum_{i=1}^{N_t-1} r_{it}r_{i+1,t}, \]  

where there are \( N_t \) daily returns, \( r_{it} \), in month \( t \). We do not subtract the sample mean from each daily return in calculating the variance.  

\[ \text{We also tried several modifications of (2), including (a) subtracting the within-month mean return from each observation, and (b) ignoring the crossproducts. These modifications had little effect on our results.} \]
Figure 1a contains a plot of the monthly standard deviation estimates for 1928-84. As Officer (1973) notes, the volatility of stock returns is larger in the 1929-40 period than it is either before or after that period. The plot in Figure 1a is not as smooth as plots of 12-month rolling estimates in Officer (1973) because each point is based on a nonoverlapping sample of returns. This plot highlights the variation in estimated volatility.

As suggested by Figure 1a, the mean and standard deviation of the stock market standard deviation estimates in Panel A of Table 1 are higher in 1928-52 than in 1953-84. The autocorrelations of $\sigma_{mt}$ in Table 1A are large and decay slowly beyond lag 3. This behavior is typical of a nonstationary integrated moving average process [see Wichern (1973)]. The standard deviation estimates are positively skewed. To adjust for this skewness we examine the logarithm of $\sigma_{mt}$. Because nonstationarity is suggested by the autocorrelations in Panel A, we examine the changes in the logarithm of the standard deviation estimates in Panel B of Table 1. The autocorrelations in Panel B are close to zero beyond lag 3. These autocorrelations suggest that the first differences of $\ln \sigma_{mt}$ follow a third order moving average process.

Panel C of Table 1 contains estimates of the moving average process,

$$ (1 - L) \ln \sigma_{mt} = \theta_0 + (1 - \theta_1 L - \theta_2 L^2 - \theta_3 L^3) u_t , $$

for 1928-84, 1928-52, and 1953-1984. The estimates of the constant term $\theta_0$ are small relative to their standard errors, suggesting that there is no deterministic drift in the standard deviation of the stock market return. The moving average estimate at lag one is large in all periods, while the estimate at lag two is largest in the first subperiod and the estimate at lag three is largest in the second. Nevertheless, the F-statistic testing the hypothesis that the model parameters are the same in 1928-52 and 1953-84 is below the .10 critical value. The small Box-Pierce statistics, $Q(12)$, support the
Figure 1a

Monthly Percent Standard Deviations of the Returns to the Standard & Poor's Composite Portfolio, $\sigma_{mt}$, Estimated from Returns for Days $i$ Within the Month $t$, $r_{it}$, 1928-84

$$
\sigma_{mt} = \left( \sum_{i=1}^{N_t} (r_{it} - \bar{r}_t)^2 + 2 \sum_{i=1}^{N_t} \sum_{i+1}^{N_t} (r_{it} - \bar{r}_t)(r_{i+1,t} - \bar{r}_t) \right)^{1/2}
$$

(2)

Figure 1b

Predicted Monthly Percent Standard Deviations of the Returns to the Standard & Poor's Composite Portfolio, $\theta_{mt}$, Estimated from the ARIMA Model in Table 1C, 1928-84

$$(1-L) \ln \sigma_{mt} = \theta_0 + (1-\theta_1 L - \theta_2 L^2 - \theta_3 L^3) u_t$$

(3)

$$\theta_{mt} = \exp \left( \ln \sigma_{mt} + .5 V(u_t) \right)$$

(4a)
### Table 1

**Time Series Properties of Estimates of the Standard Deviation of the Return to the Standard & Poor’s Composite Portfolio**

<table>
<thead>
<tr>
<th>Period</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Skewness</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>Std Error</th>
<th>Q(12)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1928-84</td>
<td>0.0474</td>
<td>0.0325</td>
<td>2.80*</td>
<td>0.71</td>
<td>0.59</td>
<td>0.55</td>
<td>0.54</td>
<td>0.50</td>
<td>0.51</td>
<td>0.54</td>
<td>0.54</td>
<td>0.49</td>
<td>0.51</td>
<td>0.49</td>
<td>0.45</td>
<td>0.04</td>
<td>2409.2*</td>
</tr>
<tr>
<td>1928-52</td>
<td>0.0607</td>
<td>0.0417</td>
<td>2.08*</td>
<td>0.68</td>
<td>0.53</td>
<td>0.50</td>
<td>0.49</td>
<td>0.43</td>
<td>0.46</td>
<td>0.51</td>
<td>0.51</td>
<td>0.44</td>
<td>0.46</td>
<td>0.44</td>
<td>0.40</td>
<td>0.06</td>
<td>893.4*</td>
</tr>
<tr>
<td>1953-84</td>
<td>0.0371</td>
<td>0.0168</td>
<td>1.70*</td>
<td>0.62</td>
<td>0.49</td>
<td>0.38</td>
<td>0.34</td>
<td>0.31</td>
<td>0.27</td>
<td>0.24</td>
<td>0.23</td>
<td>0.25</td>
<td>0.25</td>
<td>0.21</td>
<td>0.17</td>
<td>0.05</td>
<td>533.6*</td>
</tr>
</tbody>
</table>

#### A. Monthly Standard Deviation of S&P Composite Returns Estimated from Daily Data

- **Nonthly Standard Deviation of S&P Composite Return** estimated from daily data.

- **Period Hean Deviation** is calculated from the daily rates of return to the Standard & Poor’s composite portfolio for each day in the month.

\[
\sigma_{mt} = \frac{1}{N_t} \sum_{i=1}^{N_t} r_{it}^2 + 2 \sum_{i=1}^{N_t-1} r_{it} r_{i+1, t}
\]

where \( r_{it} \) is the return on day \( t \) within month \( t \), and there are \( N_t \) days in the month. **Q(12)** is the Box-Pierce [1970] statistic for 12 lags of the autocorrelation function, and **SR(ut)** is the studentized range, the sample range divided by the standard deviation. See Fama [1976, Chapter II] for a discussion and fractiles of SR under the hypothesis of a stationary normal distribution.

- **Standard errors are in parentheses.**

#### B. Percent Changes of Monthly Standard Deviation of S&P Composite Returns Estimated from Daily Data

- **Percent Change** of the monthly standard deviation estimator \( \sigma_{mt} \) is calculated from the daily rates of return to the Standard & Poor’s composite portfolio for each day in the month.

#### C. ARIMA Models for the Logarithm of the Monthly Standard Deviation of S&P Composite Returns Estimated from Daily Data

\[
(1-L) \ln \sigma_{mt} = \theta_0 + \left( 1 - \theta_1 L - \theta_2 L^2 - \theta_3 L^3 \right) u_t
\]

- **Period**

- **\( \theta_0 \), \( \theta_1 \), \( \theta_2 \), \( \theta_3 \)**

- **Skewness**

- **Box-Pierce (1970) statistic** for 12 lags of the autocorrelation function, and **SR(ut)** is the studentized range, the sample range divided by the standard deviation.

- **F-test for Stability**

Note: The monthly standard deviation estimator \( \sigma_{mt} \) is calculated from the daily rates of return to the Standard & Poor’s composite portfolio for each day in the month.

- **Greater than the .95 fractile of the sampling distribution under the hypothesis of a stationary, serially uncorrelated normal distribution.**

- **Equal to .142, .125, and .094, for \( T = 299, 384 \) and 683.**

- **Standard errors are in parentheses.**

- **The -test for stability of the time series models is based on the residual sums of squares from the subperiods and for the overall sample period, so the F-statistic would have \( k \) and \( (T-2k) \) degrees of freedom in large samples, where \( T=683 \) is the overall sample size and \( k=4 \) is the number of parameters including the constant. The value in parentheses adjusts the F-statistic for the fact that the residual variances are unequal in the two subperiods.**
hypothesis that the forecast errors from these models are random.

The skewness coefficients are small in Panels B and C of Table 1, indicating that the logarithmic transformation has removed most of the positive skewness in $\sigma_{mt}$. The studentized range statistics in Panel C are large in the overall sample period and in the first subperiod, but not in the second subperiod. The standard deviation of the errors $S(u_t)$ is about one third larger in 1928-52 than in 1953-84, which accounts for part of the large studentized range statistic for the combined sample.

We construct conditional forecasts of the standard deviation and variance of S&P returns using the formulas

$$\hat{\sigma}_{mt} = \exp \left[ \ln \hat{\sigma}_{mt} + .5 V(u_t) \right]$$

and

$$\hat{\sigma}^2_{mt} = \exp \left[ 2 \ln \hat{\sigma}_{mt} + 2 V(u_t) \right],$$

where $\ln \hat{\sigma}_{mt}$ is the fitted value for $\ln \sigma_{mt}$ from (3) and $V(u_t)$ is the variance of the prediction errors from (3) for 1928-84. If the errors $u_t$ are normally distributed, $\sigma_{mt}$ is lognormal and the corrections in (4a) and (4b) are exact. Figure 1b contains a plot of the predictions $\hat{\sigma}_{mt}$ from (4a). The predicted standard deviations track the actual standard deviations closely, although the predicted series is smoother.

The evidence in Table 1 indicates that there is substantial variation in stock market volatility. The time series models are stable over time, and the residuals appear to be random. In the subsequent tests, we interpret the transformed fitted value from these models, $\hat{\sigma}_{mt}^P$, as the predictable volatility of stock returns and the unexpected volatility, $\sigma_{mt}^{pu} = \sigma_{mt}^P - \hat{\sigma}_{mt}^P$, as proportional to the change in predicted volatility. The models seem to be stable, so we treat the parameters as if they were known to investors and we
estimate them using all the data. Conditional on the parameters, the forecasts depend only on past data.

2.2 ARCH Models

Engle (1982) proposes the autoregressive conditional heteroskedasticity (ARCH) model,

\[ r_t = \alpha + \epsilon_t, \quad \epsilon_t \sim N(0, \sigma^2_t) \] (5a)

\[ \sigma^2_t = \alpha + b \epsilon_{t-1}^2 \] (5b)

to represent a series with changing volatility. The assumption in (5b), that volatility is a deterministic function of past returns, is restrictive. For example, conditional on the time t-1 shock \( \epsilon_{t-1} \), there is no unpredictable component of volatility at time t. However, the ARCH model is attractive because the return and variance processes are estimated jointly.

We compute maximum likelihood estimates of the ARCH model using daily risk premiums, defined as the percentage change in the S&P index minus the daily yield on a one month Treasury bill, \( r_t = R_m - R_{ft} \). To account for the positive first order serial correlation in the returns to portfolios of stocks that is induced by nonsynchronous trading (see Fisher[1966] or Scholes and Williams[1977]), we generalize the model for daily risk premiums in (5a) by including a first order moving average process for the errors.

---

2 We also conducted many of our tests with one-step-ahead predictions of \( \sigma^2_{ft} \) from (3) where the parameters were estimated using the previous 60 months of data. Other variables were also used to model \( \sigma^2_{mt} \). The only variables that seem to have reliable predictive power are two lags of the return to a market portfolio, such as the CRSP value or equal weighted portfolio. Results using these alternative models are similar to the results we report.

3 Yields are calculated from the average of bid and ask prices for the U.S. government security that matures closest to the end of the month. Daily yields are calculated by dividing the monthly yield by the number of trading days in the month. These data are from the CRSP U.S. Government Securities File.

We are grateful to David Hsieh for providing the computer program used to estimate the ARCH models.
where the moving average coefficient $\theta$ will be negative. The autocorrelations of the squared risk premiums $(R_{mt} - R_{ft})^2$ decay slowly (from .27 at lag 1 to .10 at lag 60), suggesting that $\sigma_t^2$ is related to many lags of $\epsilon_t^2$. Therefore, we generalize (5b) in two alternative ways. First, we use the average of the previous 22 squared errors to predict the variance of $\epsilon_t$,

$$\sigma_t^2 = a + b \left( \sum_{i=1}^{22} \frac{\epsilon_{t-i}^2}{22} \right).$$  \hspace{1cm} (5d)

This is comparable to using the monthly variance estimates in Table 1, since there are about 22 trading days per month. Second, we use a generalized autoregressive conditional heteroskedasticity (GARCH) model (see Bollerslev [1986]) of the form

$$\sigma_t^2 = a + b \sigma_{t-1}^2 + c_1 \epsilon_{t-1}^2 + c_2 \epsilon_{t-2}^2.$$  \hspace{1cm} (5e)

Table 2 contains estimates of the ARCH model (5d) and the GARCH model (5e) for 1928-84, 1928-52 and 1953-84. The 1928-84 estimate of $b$ for the ARCH model is .96, with a standard error of .01, so there is a strong relation between recent squared errors and the estimate of volatility. The $\chi^2$ test in Table 2 implies that the parameters of the ARCH model are not equal in 1928-52 and 1953-84.

The estimates of the GARCH model (5a) and (5e) in Table 2 also indicate that the variance of daily risk premiums is highly autocorrelated. To compare the persistence implied by the GARCH model with the ARCH model (5d) it is useful to consider the sum $(b + c_1 + c_2)$, which must be less than 1.0 for the volatility process to be stationary (see Bollerslev [1986, theorem 1]). This sum equals .996, .995, and .993 for the 1928-84, 1928-52, and 1953-84 sample periods, respectively, although the $\chi^2$ test implies that the GARCH model
Table 2
ARCH Models for Daily Risk Premiums for the Standard & Poor's Composite Portfolio

\[(R_{mt} - R_{ft}) = \alpha + \varepsilon_t - \theta \varepsilon_{t-1}\]  
(5c)

\[\sigma_t^2 = a + b \sum_{i=1}^{22} \frac{\varepsilon_{t-i}^2}{22}\]  
(5d)

\[\sigma_t^2 = a + b \sigma_{t-1}^2 + c_1 \varepsilon_{t-1}^2 + c_2 \varepsilon_{t-2}^2\]  
(5e)

<table>
<thead>
<tr>
<th>ARCH Model, Equations</th>
<th>(a \times 10^3)</th>
<th>(b \times 10^5)</th>
<th>(c_1)</th>
<th>(c_2)</th>
<th>(\theta)</th>
<th>(\chi^2) test for stability</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A. January 1928 - December 1984, (T = 15,369)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ARCH (5c), (5d)</td>
<td>.287</td>
<td>1.026</td>
<td>.961</td>
<td>-1.00</td>
<td>123.2</td>
<td></td>
</tr>
<tr>
<td>( .109)</td>
<td>(.057)</td>
<td>(.013)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GARCH (5c), (5e)</td>
<td>.325</td>
<td>.075</td>
<td>.911</td>
<td>.130</td>
<td>-1.58</td>
<td></td>
</tr>
<tr>
<td>( .061)</td>
<td>(.005)</td>
<td>(.002)</td>
<td>(.007)</td>
<td>(.008)</td>
<td>96.1</td>
<td></td>
</tr>
<tr>
<td><strong>B. January 1928 - December 1952, (T = 7,326)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ARCH (5c), (5d)</td>
<td>.405</td>
<td>1.615</td>
<td>.975</td>
<td>-1.20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( .297)</td>
<td>(.105)</td>
<td>(.019)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GARCH (5c), (5e)</td>
<td>.480</td>
<td>.176</td>
<td>.888</td>
<td>.116</td>
<td>-1.091</td>
<td></td>
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<tr>
<td>( .112)</td>
<td>(.013)</td>
<td>(.004)</td>
<td>(.009)</td>
<td>(.012)</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>C. January 1953 - December 1984, (T = 8,043)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ARCH (5c), (5d)</td>
<td>.242</td>
<td>.989</td>
<td>.867</td>
<td>-1.20</td>
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</tr>
<tr>
<td>( .131)</td>
<td>(.081)</td>
<td>(.027)</td>
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<tr>
<td>GARCH (5c), (5e)</td>
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<td>.913</td>
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<td>-2.12</td>
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<tr>
<td>( .074)</td>
<td>(.007)</td>
<td>(.004)</td>
<td>(.010)</td>
<td>(.011)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notes: \((R_{mt} - R_{ft})\) is the daily risk premium for the Standard & Poor's composite portfolio (the percentage price change minus the yield on a short term default-free government bond). Nonlinear optimization techniques are used to calculate maximum likelihood estimates. Asymptotic standard errors are in parentheses under the coefficient estimates. The \(\chi^2\) test statistic is distributed \(\chi^2 \sim \chi^2_4\) for the ARCH model (5d) and \(\chi^2 \sim \chi^2_6\) for the GARCH model (5e) under the hypothesis that the parameters are equal in the two subperiods.
parameters are not equal across the two subperiods.

2.3 Stock Market Risk Premiums

We use the value-weighted portfolio of all New York Stock Exchange (NYSE) stocks from the Center for Research in Security Prices (CRSP) at the University of Chicago to measure monthly stock market returns. We use the NYSE portfolio because its returns include dividends. The S&P returns, which we use to estimate monthly variances, do not include dividends. The returns on the NYSE portfolio are highly correlated with the returns on the S&P composite portfolio. For example, the correlation between these portfolios is .993 for 1928-84. The yield on a one-month Treasury bill is subtracted from the NYSE value-weighted return to compute the risk premium.

Table 3 contains estimates of the means, standard deviations, and skewness coefficients of the monthly risk premiums. The mean risk premium is estimated in three ways: (i) using ordinary least squares (OLS); (ii) using weighted least squares, where the weight for each observation is the reciprocal of the monthly standard deviation of daily S&P returns, $1/\sigma_m$; and (iii) using weighted least squares, where the weight is the reciprocal of the predicted standard deviation from the ARIMA model in Table 1C, $1/\hat{\sigma}_m$. The WLS estimator using the actual standard deviation $\sigma_m$ gives larger estimates of the expected risk premium and larger t-statistics than either of the other estimates. This foreshadows a result in the regression tests below: in periods of unexpectedly high volatility (so that $\sigma_m$ is larger than $\hat{\sigma}_m$), realized stock returns are lower than average. These lower returns receive less weight when $1/\sigma_m$ is used to estimate the average risk premium.

As Merton (1980) stresses, variances of realized stock returns are large relative to the likely variation in expected returns. This low "signal-to-noise ratio" makes it difficult to detect variation in expected stock returns.
Table 3
Means, Standard Deviations, and Skewness of the Monthly CRSP Value-weighted Market Risk Premiums (t-statistics in parentheses)

<table>
<thead>
<tr>
<th>Period</th>
<th>Mean</th>
<th>WLS Mean&lt;sup&gt;a&lt;/sup&gt;</th>
<th>WLS Mean&lt;sup&gt;b&lt;/sup&gt;</th>
<th>Std Dev</th>
<th>Skewness</th>
</tr>
</thead>
<tbody>
<tr>
<td>1928-84</td>
<td>0.0061</td>
<td>0.0116</td>
<td>0.0055</td>
<td>0.0579</td>
<td>0.44*</td>
</tr>
<tr>
<td></td>
<td>(2.73)</td>
<td>(9.42)</td>
<td>(3.51)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1928-52</td>
<td>0.0074</td>
<td>0.0151</td>
<td>0.0083</td>
<td>0.0742</td>
<td>0.45*</td>
</tr>
<tr>
<td></td>
<td>(1.74)</td>
<td>(6.68)</td>
<td>(2.76)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1953-84</td>
<td>0.0050</td>
<td>0.0102</td>
<td>0.0044</td>
<td>0.0410</td>
<td>-0.05</td>
</tr>
<tr>
<td></td>
<td>(2.38)</td>
<td>(6.91)</td>
<td>(2.42)</td>
<td></td>
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</tr>
</tbody>
</table>

Note: The one month Treasury bill yield is subtracted from the CRSP value-weighted stock market return to create a risk premium.

<sup>a</sup> Sample mean estimated by weighted least squares, where the standard deviation of the Standard & Poor's composite portfolio estimated from the days within the month, $\sigma_{rt}$, is used to weight the observations.

<sup>b</sup> Sample mean estimated by weighted least squares, where the predicted standard deviation of the Standard & Poor's composite portfolio estimated from the ARIMA model in Table 1C is used to weight the observations.

* Greater than the .95 fractile of the sampling distribution under the hypothesis of a stationary, serially uncorrelated normal distribution.
For example, consider the average risk premiums for 1928-52 and 1953-84. The sample standard deviations are much higher in the first subperiod, and the mean risk premiums are also higher in that period. However, the standard errors of the sample means are so large that neither the hypothesis that the subperiod expected premiums are equal, nor the hypothesis that expected risk premiums in 1928-52 are twice the expected premiums in 1953-84 can be rejected at conventional significance levels. The tests below provide more structured ways to assess the relation between expected risk premiums and volatility.

3. Estimating Relations Between Risk Premiums and Volatility

3.1 Regressions of Monthly Risk Premiums on ARIMA Forecasts of Volatility

In an efficient capital market, investors use best conditional forecasts of variables, such as the standard deviation of stock returns, that affect equilibrium expected returns. Thus, we can estimate the relation between expected risk premiums and volatility by regressing realized risk premiums on the predictable components of the stock market standard deviation or variance,

\[ (R_{mt} - R_{ft}) = \alpha + \beta \hat{\sigma}_m^P + \epsilon_t. \]  

If \( \beta = 0 \) in (6), the expected risk premium is unrelated to the variability of stock returns. If \( \alpha = 0 \) and \( \beta > 0 \), the expected risk premium is proportional to the standard deviation \( (p = 1) \) or variance \( (p = 2) \) of stock returns.

Table 4 contains weighted least squares estimates of regression (6). Each observation is weighted by the predicted standard deviation \( \hat{\sigma}_m \) from the ARIMA model in Table 1C to correct for heteroskedasticity. Two sets of standard errors are calculated for each regression. The first (in parentheses) is based on the usual least squares formula. The second [in brackets] is based
Table 4

Weighted Least Squares Regressions of Monthly CRSP Value-Weighted Risk Premiums Against the Predictable and Unpredictable Components of the Standard Deviations or Variances of Stock Market Returns

\[(R_{mt} - R_{ft}) = \alpha + \beta \sigma_{mp}^2 + \epsilon_t \quad (6)\]

\[(R_{mt} - R_{ft}) = \alpha + \beta \sigma_{mp}^2 + \gamma \sigma_{mu}^2 + \epsilon_t \quad (7)\]

<table>
<thead>
<tr>
<th>Volatility Measure</th>
<th>Equation (6)</th>
<th>Equation (7)</th>
</tr>
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<tbody>
<tr>
<td></td>
<td>(\alpha)</td>
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<tr>
<td>(\sigma_{mt})</td>
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<td>.023</td>
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<tr>
<td></td>
<td>(.0043)</td>
<td>(.116)</td>
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<td>(\sigma_{mt}^2)</td>
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<td>.335</td>
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<td>(\sigma_{mt})</td>
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<td>(\sigma_{mt}^2)</td>
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<td>(.0041)</td>
<td>(.139)</td>
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<tr>
<td>(\sigma_{mt})</td>
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<td>(.0059)</td>
<td>(.182)</td>
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<td>(\sigma_{mt}^2)</td>
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</tr>
<tr>
<td></td>
<td>(.0032)</td>
<td>(2.192)</td>
</tr>
</tbody>
</table>

Note: \(\sigma_{mt}\) is the prediction and \(\sigma_{mt}^u\) is the prediction error for the estimate of the monthly stock market standard deviation from the ARIMA model in Table 1C. \(\sigma_{mt}\) and \(\sigma_{mt}^u\) are the prediction and prediction error for the variance of stock returns. The estimated time series model for \(\sigma_{mt}\) is reported in Table 1C. Standard errors are in parentheses below the coefficient estimates. The numbers in brackets [ ] are standard errors based on White's [1980] consistent heteroskedasticity correction. \(R^2\) is the coefficient of determination, \(Q(12)\) is the Box-Pierce statistic for 12 lags of the residual autocorrelation function which should be distributed as \(X^2_{12}\), and \(SR(\epsilon)\) is the studentized range of the residuals. These regressions are estimated using weighted least squares (WLS), where the predicted standard deviation of the S&P composite portfolio \(\sigma_{mt}\) is used to standardize each observation. \(R^2\), \(Q(12)\) and \(SR(\epsilon)\) are based on the weighted residuals, but the standard deviation of the residuals is based on the unweighted residuals (in the same units as the original data).
on White's (1980) consistent correction for heteroskedasticity.4

The estimates of regression (6) provide little evidence of a relation between expected risk premiums and predictable volatility. For example, the 1928-84 estimate of $\beta$ is .02, with a standard error of .12, in the standard deviation specification ($p = 1$) and .34, with a standard error of .94, in the variance specification ($p = 2$). All of the estimates of $\beta$ are within one standard error of zero.

Regressions measuring the relation between realized risk premiums and contemporaneous unexpected changes in market volatility,

$$(R_{mt} - R_{ft}) = \alpha + \beta \sigma_{mt}^P + \gamma \sigma_{mt}^{pu} + \epsilon_t,$$

provide more reliable evidence. In this regression, $\sigma_{mt}^{pu} = \sigma_{mt}^P - \sigma_{mt}^P$ is the unpredicted standard deviation ($p = 1$) or variance ($p = 2$) of returns from the ARIMA model in Table 1C. The unpredicted components of volatility are essentially uncorrelated with the predicted components, so including them in the regressions should not affect the estimates of $\beta$. However, including $\sigma_{mt}^{pu}$ improves the tests in two ways. First, by explaining more of the realized risk premiums, the standard errors of the regression coefficients are reduced. More important, the coefficient on the unpredicted component of volatility $\gamma$ provides indirect evidence about the effects of predictable volatility on ex ante risk premiums.

4The importance of correcting for heteroskedasticity is illustrated by results in Gennotte and Marsh (1985). They estimate a model like (6), except the prediction of this month's variance is the square of last month's risk premium. Their estimate of $\beta$ is more than five standard errors from zero for 1926-78 using the CRSP equal weighted portfolio. We replicated their estimates for 1928-84, and the OLS estimate of $\beta$ is .69 with a standard error of .11. However, the regression errors are heteroskedastic, much like the behavior of the standard deviation of market returns in Figure 1a. The OLS standard errors are too small, since White's (1980) corrected standard error is .42. The WLS estimate of $\beta$ is .75 with a standard error of .25 (.29 with White's correction). Thus, the reliability of the relation reported by Gennotte and Marsh (1985) is overstated because of heteroskedasticity.
Suppose this month's standard deviation is larger than predicted. Then the model in Table 1C implies that all forecasted standard deviations will be revised upward. If the risk premium is positively related to the predicted standard deviation, the discount rate for future cash flows will also increase. If the cash flows are unaffected, the higher discount rate reduces both their present value and the current stock price.\(^5\) Thus, a positive relation between the predicted stock market volatility and the expected risk premium induces a negative relation between the unpredicted component of volatility and the realized risk premium.

Table 4 contains WLS estimates of (7). There is a reliably negative relation between realized risk premiums and unpredicted changes in the volatility of stock returns. The estimated coefficients of the unexpected change in the standard deviation \(\gamma\) range from -1.01 to -1.04, with t-statistics between -6.88 and -10.98. The estimates of \(\gamma\) in the variance specification vary from -3.99 to -9.08, with t's between -5.78 and -8.95.

Again, regression (7) provides little direct evidence of a relation between the expected risk premium and volatility. Five of the six estimates of \(\beta\) are negative, and only one is more than one standard error from zero.\(^6\)

Many of the estimates of \(\alpha\) are reliably positive in (7). For example, the estimate for 1928-84 is .0077 with a standard error of .0039 when \(p = 1\), and .0057 with a standard error of .0020 when \(p = 2\). This implies that the expected risk premium is not proportional to either the predicted standard

\(^5\)This volatility-induced change in the stock price in turn contributes to the volatility estimated for that month, but this effect is likely to be a negligible fraction of the month's total unexpected volatility.

\(^6\)The correlation between the predicted standard deviation from the ARIMA model \(\delta_{mt}\) and the prediction error \(\sigma_u\) is -.07 for the 1928-84 sample period. This small negative correlation, and the highly significant negative coefficient on the prediction error, causes a number of the estimates of \(\beta\) to change sign relative to the simple regression (6).
deviation or the predicted variance of stock market return. It also implies that the conditional expected slope of the capital market line,

\[ E_{t-1}[(R_{mt} - R_{ft})/\hat{\sigma}_{mt}], \]

is not constant.

The evidence in Table 4 provides little basis to choose between the standard deviation and variance specifications of the relation between volatility and expected risk premiums. The residual variances \( S(\epsilon) \) are smaller for the estimates of the standard deviation specification in (7), and the \( R^2 \) statistics based on weighted residuals are larger (except for 1928-52). However, these differences favoring the standard deviation specification are not large. 7

3.2 GARCH-in-Mean Models

Engle, Lilien and Robins (1984) and Bollerslev, Engle and Wooldridge (1985) propose generalizations of the ARCH model that allow the conditional mean return to be a function of volatility. Table 5 contains estimates of the GARCH-in-mean model in two forms:

\[ (R_{mt} - R_{ft}) = \alpha + \beta \sigma_t + \epsilon_t - \theta \epsilon_{t-1} \]  
\[ (R_{mt} - R_{ft}) = \alpha + \beta \sigma_t^2 + \epsilon_t - \theta \epsilon_{t-1}, \]

where \( R_{mt} - R_{ft} \) is the daily risk premium on the S&P composite portfolio, and \( \sigma_t^2 \), the variance of the unexpected risk premium \( \epsilon_t \), follows the process in (5e). As before [cf. (5c)], the moving average term \( \theta \epsilon_{t-1} \) is included to capture the effect of nonsynchronous trading. The slope has the same

---

7 Pagan (1984) and Murphy and Topel (1985) argue that regressions with generated regressors (such as (6) or (7)) produce understated standard errors because the randomness in the predictions is ignored. Following Pagan, we estimated (6) using several lags of \( \sigma_{mt} \) as instrumental variables, and the estimates of \( \beta \) and their standard errors were similar to the estimates in Table 4. We also calculated the adjustment suggested by Murphy and Topel (1985). The results in Table 4 were unaffected, so they are not reported. Pagan and Ullah (1985) discuss other estimation strategies, including ARCH models, for models like (6) or (7).
Table 5
GARCH-in-mean Models for Daily Risk Premiums for the Standard & Poor's Composite Portfolio

\[
(R_{mt} - R_{ft}) = \alpha + \beta \sigma_t^2 + \varepsilon_t - \delta \varepsilon_{t-1} \quad (8a)
\]

\[
(R_{mt} - R_{ft}) = \alpha + \beta \sigma_t + \varepsilon_t - \delta \varepsilon_{t-1} \quad (8b)
\]

\[
\sigma_t^2 = a + b \sigma_{t-1}^2 + c_1 \varepsilon_{t-1}^2 + c_2 \varepsilon_{t-2}^2 \quad (5e)
\]

<table>
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<th>(\beta)</th>
<th>(a \times 10^5)</th>
<th>(b)</th>
<th>(c_1)</th>
<th>(c_2)</th>
<th>(\delta)</th>
<th>(\chi^2) test for stability</th>
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<td>.077</td>
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<td>(.005)</td>
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<td>.888</td>
<td>.115</td>
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<td>(.011)</td>
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<td>(.007)</td>
<td>(.005)</td>
<td>(.011)</td>
<td>(.011)</td>
<td>(.012)</td>
<td></td>
</tr>
</tbody>
</table>

Notes: \((R_{mt} - R_{ft})\) is the daily risk premium for the Standard & Poor's composite portfolio (the percentage price change minus the yield on a short term default-free government bond). Nonlinear optimization techniques are used to calculate maximum likelihood estimates. Asymptotic standard errors are in parentheses under the coefficient estimates. The \(\chi^2\) statistic is distributed \(\chi^2_7\) under the hypothesis that the parameters are equal in the two subperiods.
interpretation in (8b) that it has in the monthly ARIMA regression (6) with $p = 2$, because both the risk premium and the variance $\sigma^2_t$ should be approximately proportional to the length of the measurement interval. Since the standard deviation $\sigma_t$ should be proportional to the square root of the measurement interval, the estimate of $\beta$ in (8a) should be about 4.5 times smaller than the comparable monthly estimate in (6) with $p = 1$. The intercept $\alpha$ has the dimension of an average daily risk premium in (8a) and (8b), so it should be about 22 times smaller than the monthly estimates in (6).

The results in Table 5 indicate there is a reliably positive relation between expected risk premiums and the predicted volatility. The estimated coefficient of predicted volatility $\beta$ for 1928-84 is .077, with a standard error of .022, in the standard deviation specification (8a), and 2.521, with a standard error of .917, in the variance specification (8b). This evidence supports our interpretation of the negative relation between realized risk premiums and the unexpected change in volatility in Table 4.

As with the results in Table 4, the standard deviation specification (8a) of the GARCH model fits the data slightly better than the variance specification (8b). The log-likelihoods from the GARCH model are larger for the standard deviation specification in 1928-84 and 1928-52. Also, if the power of the standard deviation is unconstrained when the model is estimated,

$$\left(R_{mt} - R_{ft}\right) = \alpha + \beta \sigma^2_t + \epsilon_t - \theta \epsilon_{t-1},$$

the estimates of $p$ are closer to 1.0 than 2.0. However, the standard errors of the estimates are large. The evidence in favor of the standard deviation

---

If the errors in (8a) are serially independent, the variance of the $N_t$-step-ahead forecast error (ignoring parameter estimation error) is $N_t \sigma^2_t$. Since the variance process in (5e) is almost a random walk, the sum of the one through $N_t$-step-ahead forecasts of the risk premium is approximately $N_t E(R_{mt} - R_{ft} | \sigma_t)$. 13
specification is not strong.

3.3 Comparisons of ARIMA and GARCH Models

The ARIMA models in Table 4, which use monthly risk premium data, and the GARCH-in-mean models in Table 5, which use daily data, yield sufficiently different results that it is worth exploring the relation between these models. Table 6A contains estimates of the GARCH-in-mean models in (8a) and (8b) using monthly risk premiums. The estimates in Table 6A do not use daily return data to predict the volatility of risk premiums, so one would expect the volatility estimates to be less precise. Nevertheless, the estimates of $\beta$, the coefficient of predicted volatility, are quite large in comparison with the regression model estimates in Table 4. In particular, the estimates of $\beta$ for 1928-84 and 1953-84 in Table 6A are closer to the estimates from the comparable daily GARCH-in-mean models in Table 5 than to the regression estimates in Table 4, and several are more than two standard errors above zero.

Table 6B contains estimates of the regression of the monthly risk premium on the prediction of the monthly standard deviation or variance from the monthly GARCH-in-mean model in Table 6A,

$$
(R_{mt} - R_{ft}) = \alpha + \beta \sigma_t + \epsilon_t
$$

(10a)

and

$$
(R_{mt} - R_{ft}) = \alpha + \beta \sigma_t^2 + \epsilon_t
$$

(10b)

Each observation in these regressions is weighted by the predicted monthly standard deviation $\sigma_t$ from Table 6A. Table 6B contains estimates of (10a) and (10b) that are comparable to the estimates of regression (6) in Table 4. The

---

9 Note that the daily estimate of $\beta$ for the standard deviation specification (10a) in Table 5 should be multiplied by the square root of the number of days in the month, $\sqrt{N_t} = 4.5$, to be comparable to the monthly estimates in Table 6A. Thus, the values of $\beta$ for (10a) implied by Table 5 are .30, .25, and .39, for 1928-84, 1928-52, and 1953-84, respectively.

14
Table 6

Comparison of ARIMA
with GARCH Predictions of Stock Market
Volatility and their Relations
to Monthly CRSP Value-weighted Risk Premiums

A. GARCH-in-mean Estimates Using

Monthly Risk Premiums

\[
(R_{mt} - R_{ct}) = \alpha + \delta R_{ct-1} + \epsilon_t \quad \text{(8a)}
\]

\[
(R_{mt} - R_{ct}) = \alpha + \delta R_{ct-1} + \epsilon_t \quad \text{(8b)}
\]

\[
\sigma_t^2 = \alpha + \beta \epsilon_{t-1}^2 + \gamma \epsilon_{t-1}^2 + \epsilon_t^2 \quad \text{(8c)}
\]

<table>
<thead>
<tr>
<th>Sample Period</th>
<th>Garch-in-mean Equations</th>
<th>(a)</th>
<th>(\beta)</th>
<th>(c_1)</th>
<th>(c_2)</th>
<th>(b)</th>
<th>(y^2) test for stability of (\delta)</th>
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<tbody>
<tr>
<td>2/28 - 12/84</td>
<td>Standard Deviation</td>
<td>0.0003</td>
<td>0.178</td>
<td>0.164</td>
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<td></td>
<td>(8a), (8e)</td>
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<td>(0.134)</td>
<td>(0.030)</td>
<td>(0.028)</td>
<td>(0.047)</td>
<td>(0.056)</td>
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<tr>
<td></td>
<td>Variance</td>
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<td>1.855</td>
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<td>0.767</td>
<td>0.066</td>
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<tr>
<td></td>
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<td>(0.056)</td>
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<td>(3.567)</td>
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<td>(0.060)</td>
<td>(0.046)</td>
<td>(0.069)</td>
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B. Weighted Least Squares Regressions of Monthly CRSP Value-weighted Risk Premiums

Against the Predicted Standard Deviation or Variance of Stock

Returns from the Monthly GARCH Model

\[
(R_{mt} - R_{ct}) = \alpha + \delta R_{ct-1} \quad \text{(10a)}
\]

\[
(R_{mt} - R_{ct}) = \alpha + \delta R_{ct-1} + \epsilon_t^2 \quad \text{(10b)}
\]

<table>
<thead>
<tr>
<th>Sample Period</th>
<th>Volatility Measure</th>
<th>(a)</th>
<th>(\delta)</th>
<th>(r(\epsilon))</th>
<th>(X^2)</th>
<th>(Q(12))</th>
<th>(SR(\epsilon))</th>
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<td>2/28 - 12/84</td>
<td>Monthly GARCH</td>
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<tr>
<td></td>
<td>Standard Deviation</td>
<td>(0.0063)</td>
<td>(0.142)</td>
<td>(0.168)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>(0.0007)</td>
<td>(1.056)</td>
<td>(0.118)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2/28 - 12/52</td>
<td>Monthly GARCH</td>
<td>0.0233</td>
<td>-0.279</td>
<td>0.0751</td>
<td>0.022</td>
<td>11.7</td>
<td>7.31</td>
</tr>
<tr>
<td></td>
<td>Standard Deviation</td>
<td>(0.0091)</td>
<td>(0.184)</td>
<td>(0.233)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>(0.0027)</td>
<td>(1.256)</td>
<td>(1.304)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/53 - 12/84</td>
<td>Monthly GARCH</td>
<td>-0.0126</td>
<td>0.400</td>
<td>0.0408</td>
<td>0.004</td>
<td>16.9</td>
<td>5.91</td>
</tr>
<tr>
<td></td>
<td>Standard Deviation</td>
<td>(0.0107)</td>
<td>(0.260)</td>
<td>(0.237)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Variance</td>
<td>(0.0049)</td>
<td>(2.755)</td>
<td>(2.946)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: See footnotes to Tables 2 and 4 for a description of the statistical procedures.

The \(X^2\) statistic is distributed \(X^2\) under the hypothesis that the parameters are equal in the two subperiods.
estimates of the coefficient of predicted volatility $\beta$ are very small relative to their standard errors in the 1928-84 sample period. In the 1928-52 period the estimates of $\beta$ are negative, while in the 1953-84 period they are positive, although none of the estimates is more than two standard errors from zero. Although these regressions use the GARCH-in-mean estimates of predicted volatilities, they provide no evidence of a relation between expected risk premiums and predictable volatility.

As a final comparison of the regression and GARCH-in-mean models, we create a series of monthly predicted standard deviations from the daily GARCH-in-mean model in Table 5 by using the fitted GARCH process (5e) to forecast $\sigma_t^2$ for each of the $N_t$ trading days in the month, conditional on data available as of the first day in the month. We compute the implied monthly standard deviation by summing the predicted variances within the month and taking the square root of the sum. We estimate the expected monthly risk premium from the GARCH-in-mean model by inserting the predicted standard deviations for the days in the month into (8a) and summing the predicted daily expected risk premiums.

The GARCH-in-mean prediction of the monthly standard deviation is similar to the ARIMA prediction $\hat{\sigma}_{mt}$ (the correlation for 1928-84 is .89 and the means are virtually identical). The GARCH-in-mean and ARIMA predictions have essentially the same correlation with the actual monthly standard deviation $\sigma_{mt}$ (.755 and .744), although the sample variance of the GARCH prediction is about one third larger. Thus, the two models have similar abilities to predict future volatility.  

In contrast, the behavior of the expected risk premiums implied by the regression and GARCH-in-mean models is quite different. Figure 2a contains a
plot of the monthly expected risk premium from regression (6) with p=1 for 1928-84 from Table 4, and Figure 2b contains a plot of the monthly expected risk premium from the daily GARCH-in-mean model (8a). The correlation between the two measures is .88 over the full sample period. However, the predicted risk premiums from the daily GARCH model have a much higher mean and variance than the predictions from regression (6). (Note that the scale in Figure 2a is from 0.5 to 1.0 percent per month, and the scale in Figure 2b is from 0 to 9 percent per month.) The higher variability of predicted risk premiums in Figure 2b is caused by two factors: (a) the greater variability of the predicted standard deviation from the GARCH-in-mean model, and (b) the larger coefficient of the predicted standard deviation, \( \beta \), in the GARCH-in-mean model. (The sensitivity of the monthly expected risk premium to a change in the predicted monthly standard deviation is about 15 times greater in the GARCH-in-mean model than in the ARIMA regression model.) Thus, although the ARIMA and GARCH-in-mean models have similar abilities to predict volatility, the GARCH-in-mean model implies greater variability of expected risk premiums.\(^1\) More puzzling, however, is the fact that the average of the expected risk premiums is much higher for the GARCH-in-mean model (.0134 versus .0058 for the regression model on ARIMA predictions of the standard deviation). The GARCH-in-mean predictions seem too high, since they are almost twice the average realized premium (see Table 3).

The high GARCH-in-mean predictions probably reflect the negative relation

\[^1\]The greater variability of the expected risk premiums from the GARCH-in-mean model does not arise solely because we use forecasts conditional on information for the first day in the month to construct each month's forecast. We also constructed monthly "forecasts" by cumulating all of the one-step-ahead daily forecasts within each month. The variance of these forecasts is .000872, while the variance of the monthly forecasts constructed from first-of-the-month estimates is .000867.
Predicted Monthly Risk Premium to the Standard & Poor's Composite Portfolio from the Regression on ARIMA Predictions of the Standard Deviation, \( \sigma_{mt} \), in Table 4, 1928-84

\[
(R_{mt} - R_{ft}) = \alpha + \beta \sigma_{mt}
\]  

(6)

Predicted Monthly Risk Premium to the Standard & Poor's Composite Portfolio from the Daily GARCH-in-mean model for the Standard Deviation, \( \sigma_t \), in Table 5, 1928-84

\[
(R_{mt} - R_{ft}) = \alpha + \beta \sigma_t - \theta c_{t-1}
\]  

(8a)

\[
\sigma_t = \omega + \beta \sigma^2_{t-1} + \gamma c^2_{t-1} + \delta c^2_{t-2}
\]  

(5e)
between the unexpected component of volatility and the unexpected return observed in Table 4. The likelihood function used to estimate the GARCH model assumes that the standardized residuals $\epsilon_t/\sigma_t$ have a unit normal distribution. The daily standardized residuals from the standard deviation specification (8a) and (5e) in Table 5 have a mean of -.038, a standard deviation of .999, and a skewness coefficient of -.37. Since $\sigma_t$ is predetermined in the GARCH model, the negative skewness of the standardized residuals reflects the negative relation between the unexpected component of volatility and the unexpected risk premium. Negative skewness probably causes the negative mean of the standardized residuals. This violation of the normality assumption may cause the high GARCH-in-mean predictions of the risk premiums (e.g., the average monthly error from the GARCH-in-mean model is -.0090, which is greater than the difference between the average predicted risk premiums in Figures 2a and 2b). Of course, this argument says that the level of the predictions in Figure 2b is too high because $\alpha$ is too large; it is also possible that the sensitivity of expected risk premiums to changes in predictable volatility $\beta$ is biased.

It is well known in the econometrics literature that full information maximum likelihood (FIML) estimators, such as the GARCH models, are more efficient than instrumental variables estimators, such as the two-step regression procedures, although both estimators are consistent if the model is correctly specified. On the other hand, FIML estimators are generally more sensitive to model misspecification than instrumental variables estimators. Hausman[1978] proposes a class of model specification tests based on this

\[ 12 \text{Indeed, a WLS regression of the daily errors from (8a) } \epsilon_t \text{ on the } \ "\text{unexpected}\" \text{ standard deviation, } [\epsilon_t - \sigma_t], \text{ yields a coefficient of -.375 with a } t \text{-statistic of -34.1. This regression is similar to the multiple regression estimates of equation (7) in Table 4.} \]
observation. Thus, one interpretation of the apparent differences between the
GARCH-in-mean and the regression results is that the statistical specification
underlying these models is not adequate. A formal test such as Hausman's is
difficult because the GARCH and ARIMA models for volatility are not nested.
It is likely that neither model is entirely adequate for predicting expected
risk premiums.

4. Analysis of the Results

4.1 Interpreting the Estimated Coefficients

Merton (1980) notes that in a model of capital market equilibrium where a
"representative investor" has constant relative risk aversion, there are
conditions under which the expected market risk premium will be approximately
proportional to the ex ante variance of the market return,\textsuperscript{13}

$$E_{t-1}(R_{mt} - R_{ft}) = C \sigma^2_{mt}. \quad (11)$$

The parameter $C$ in (11) is the representative investor's coefficient of
relative risk aversion. For example, the logarithmic utility function for
wealth, $U(W) = \log W$, implies $C = 1$. If we ignore the intercepts ($a$), the
coefficient of relative risk aversion equals $\beta$ in both the regression model
(6) for $p = 2$ and in the GARCH-in-mean model (8b).

The estimate of relative risk aversion ($\beta$) from the regression model is .34
for the overall period, but the large standard error (.90) does not allow us
to reliably distinguish the coefficient from zero. The corresponding GARCH-
in-mean estimate of $\beta$ is 2.52, which is about 2.75 times its estimated

\textsuperscript{13} This approximation will hold in Merton's (1973) intertemporal model if
(a) the partial derivative of the representative investor's consumption with
respect to wealth is much larger than the partial derivative with respect to
any state variable or (b) the variance of the change in wealth is much larger
than the variance of the change in any state variable.
standard error. Both of these point estimates appear economically reasonable, however, and they are well within the range of estimates produced by other studies using different approaches. For example, Friend and Blume (1975) estimate relative risk aversion to be about 2.0, Hansen and Singleton (1982) obtain estimates between -1.6 and 1.6, and Brown and Gibbons (1985) obtain estimates between 0.1 and 7.3. Estimates of relative risk aversion can also be obtained from the standard deviation specifications in (6) for \( p = 1 \) and in (8a) by dividing the \( \beta \) estimates by an average value of \( \sigma_m \). The estimates obtained in this fashion are similar to those obtained from the variance specifications.

As noted above, a negative ex post relation between risk premiums and the unexpected component of volatility is consistent with a positive ex ante relation between risk premiums and volatility. The negative coefficient in the ex post relation is also likely to be greater in magnitude than the positive coefficient in the ex ante relation, especially when volatilities, and thus expected future risk premiums, are highly autocorrelated. This can be seen easily using a model developed by Poterba and Summers (1986). They model volatility as a first order autoregressive process to illustrate this effect. Let \( \rho \) denote the first-order autocorrelation of the variance. Assume that expected real dividends grow at a constant rate \( g \), that the real risk-free rate is a constant \( r_f \), and that the expected risk premium in period \( t+\tau \), conditional on \( \sigma_{t+\tau}^2 \), equals \( \beta \mathbb{E}(\sigma_{t+\tau}^2 | \sigma_t^2) \). Then, as Poterba and Summers show, the percentage change in stock price arising from a change in volatility is approximately
The quantity in brackets in (12) exceeds unity and is increasing in \( \rho \). The value of the derivative is sensitive to the choice of \( \rho \), but an example can illustrate the potential magnitude of the ex post relation between returns and volatility relative to the ex ante relation. Assume that (i) the monthly variance \( \sigma^2_t \) equals .002, (ii) the real risk-free rate equals .035 percent per month, and (iii) real dividends are expected to grow at .087 percent per month. (The last two values are the same as those used by Poterba and Summers.) The estimate of the coefficient of unpredicted component of volatility for 1928-84 in Table 4 is -4.438, which implies that \( \beta = 2.05 \) if \( \rho = .5 \), and \( \beta = 1.07 \) if \( \rho = .7 \). Given these hypothetical magnitudes, and the estimated standard errors in Table 4, it is not surprising that an ex post negative relation is detected more strongly in the data than is an ex ante positive relation.\(^{14}\)

4.2 The Effect of Leverage

Many of the firms whose common stocks constitute the indexes used in computing the market risk premiums are levered. While the observed strong negative relation between risk premiums and unexpected volatility is consistent with a positive ex ante relation between risk premiums and

\[
\frac{d \log P}{d \sigma^2} = - \left[ \frac{1}{1 + r_f + \beta \sigma^2_t - \rho (1 + g)} \right] \beta \quad . (12)
\]

\(^{14}\)Poterba and Summers estimate that the elasticity of the stock price with respect to the variance \( \sigma^2_t \) is about 10 times higher for the IMA(1,3) model in Table 1C than for their AR(1) model. This means that the implied values of \( \beta \) would be correspondingly smaller, given our estimates of \( \gamma \). In an earlier version of this paper we presented estimates of \( \beta \) that used a constraint similar to (12). The constrained estimates of \( \beta \) were small, but several standard errors from zero, reflecting the precision of the estimates of \( \gamma \).
volatility at the firm level, Black (1976) and Christie (1982) suggest another interpretation. They note that leverage can induce a negative ex post relation between returns and volatility for common stocks, even if the volatility and the expected return for the total firm are constant.

Suppose a firm's volatility and expected return are constant. A decline in stock prices (relative to bond prices) increases leverage, increases the expected return on the stock, and increases the variance of the stock's return. As Black (1976) and Christie (1982) demonstrate, however, if this is the sole reason for the relation between stock returns and volatility, then a regression of the percent change in standard deviation on the percent change in stock price should have a coefficient (elasticity) greater than -1.0.

An elasticity of -1.0 is an extreme lower bound. Consider a firm with riskless debt. The elasticity of the stock return standard deviation with respect to the stock price is \(-D/V\), where \(D\) is the value of the debt and \(V\) is the value of the firm. The lower bound of -1.0 only occurs when the stock has no value. Evidence in Taggart (1983) suggests that the fraction of debt in the capital structure of large U.S. corporations is below 45 percent throughout 1926-79, so the leverage hypothesis should not generate an elasticity below -0.45.

To test the hypothesis that the relation between realized risk premiums and unexpected volatility is caused only by leverage, we regress the percentage change in the estimated standard deviation of the S&P composite portfolio against the continuously compounded return on that portfolio,

\[
\ln(\sigma_{mt}/\sigma_{mt-1}) = \alpha_0 + \alpha_1 \ln(1+R_{mt}) + \epsilon_t .
\]  

(13)

The estimated elasticity \(\alpha_1\) is -1.69, with a standard error of .25, for 1928-84. The estimates for 1928-52 and 1953-84 are -1.63 and -1.89, with
standard errors of .32 and .45. The estimated elasticity is reliably less than -1.0. Black (1976) obtains a similar result using a sample of 30 stocks from May 1964 to December 1975. Our longer sample period and more inclusive market index support Black's conclusion: leverage is probably not the sole explanation for the negative relation between stock returns and volatility.

4.3 Extensions

This paper examines the time series relation between the risk of a stock market portfolio and the portfolio's expected risk premium. The tests above use the volatility of stock returns as the conditional measure of risk. We have also tried to estimate this relation using several other measures of risk, including the variability of the real interest rate, the covariance between the stock market return and consumption, and the variability of decile portfolios formed on the basis of firm size. All of these variables involve monthly data, so none is estimated as precisely as our measure of volatility. Perhaps because of this estimation problem, none of these risk measures produces a stronger relation between risk and return than we observe using the volatility of stock returns.

We have also tried to improve the tests by including other predictive variables in the models. Fama and Schwert (1977) show that the nominal interest rate can be used to predict stock returns. Keim and Stambaugh (1985) use (i) the yield spread between long-term low grade corporate bonds and short-term Treasury bills, (ii) the level of the S&P composite index relative to its average level over the previous 45 years, and (iii) the average share price of the firms in the smallest quintile of New York Stock Exchange firms to predict stock returns. Including these variables in the models does not have much impact on our estimates of the time series relation between risk and return.
5. Conclusions

We find evidence of a positive relation between the expected risk premium on common stocks and the predictable level of volatility. However, the variability of realized stock returns is so large that it is difficult to discriminate among alternative specifications of this relation. We present several estimates of the relation between the expected risk premium and the predicted volatility of NYSE common stocks over 1928-84.

There is also a strong negative relation between the unpredictable component of stock market volatility and realized risk premiums. If expected risk premiums are positively related to predictable volatility, then a positive unexpected change in volatility (and an upward revision in predicted volatility) increases future expected risk premiums and lowers current stock prices. The magnitude of the negative relation between contemporaneous returns and changes in volatility is too large to be attributed solely to the effects of leverage discussed by Black (1976) and Christie (1982), so we interpret this negative relation as evidence of a positive relation between expected risk premiums and ex ante volatility.

The estimates of volatility and expected risk premiums in this paper suggest that there has been substantial variation in these variables over the past 60 years. While we are unwilling to choose a particular model for expected risk premiums as they relate to predictable movements in volatility, it seems obvious that future work is called for in this area. Other variables that could affect expected risk premiums should be integrated into this analysis, as well as different measures of time varying risk. We have done some work along these lines, but the results were so ambiguous that they are not worth reporting in detail, probably because the measures of risk and other factors that might affect expected risk premiums are less precise than the
volatility measures reported above.
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