Efficient investment in children
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Environment
Discrete time infinite horizon economy with periods denoted by \( t \in \{0,1,2,3,\ldots\} \). In each period there is a \([0,1]\) continuum of adults and a \([0,1]\) continuum of children. Adults in period \( t \) die at the end of period \( t \) and children in period \( t \) become adults in period \( t+1 \) and a new generation of children is born in period \( t+1 \), and so on. Let \( F(a) \) be the distribution of children according to innate ability, where we take \( F \) as a primitive. At this point we assume that the ability of a child in period \( t \) is perfectly known in period \( t \). (An alternative assumption one could explore is to assume that it is unobserved in period \( t \).) Each adult has one unit of time which may be used for producing goods or for caring for children. Adults differ according to their productivities. Let \( G_t(\pi) \) be the distribution of adults according to productivity in period \( t \). \( G_0 \) is a given initial condition while future \( G \)'s will be determined endogenously as we will see below. Let \( n_t(\pi) \) be the adult time spent in production in period \( t \). The total output of goods in period \( t \) may be consumed or invested in children. Let \( c_t(a) \) be the amount of goods invested in period \( t \) in a child of ability \( a \). Let \( T_t(a) \) be the adult time spent in period \( t \) on a child of ability \( a \). The future productivity of a child is then determined by the following relation.

\[
(1) \quad \pi_{t+1}(a) = Q(a,T_t(a),c_t(a)).
\]

The function \( Q \) is taken as a primitive and we assume that \( Q \) is strictly increasing in all its arguments.

Note that the productivity distributions evolve as follows.

\[
(2) \quad G_{t+1}(\pi) = m\{s:Q(a,T_t(a),c_t(a)) \leq \pi\},
\]

where \( m \) is the measure on the set of abilities corresponding to the distribution \( F \).

Let \( C_t \) be aggregate consumption in period \( t \). The resource constraint for this economy is as follows.

\[
(3) \quad C_t + \int c_t(a)dF(a) \leq \int n_t(\pi)dG_t(\pi).
\]
There is also a time constraint for this economy. The total time spent on children plus the total time spent on production must be no greater than the total time endowment which is unity. This implies the following.

\[(4) \quad \int n_t(\pi) dG_t(\pi) + \int T_t(a) dF(a) \leq 1.\]

**Efficiency**

By efficiency we mean that it's not possible to have more consumption at some date without having less consumption at some other date. The problem of efficient investment in children is to determine the schedules \(n_t(\pi)\), \(T_t(a)\) and \(c_t(a)\) in each period given this efficiency criterion.

**Characterizing efficient allocations**

Characterizing the schedule \(n_t(.)\) is straightforward. It's obvious that there should be some \(\omega_t\) such that

\[(5) \quad n_t(\pi) = 1 \text{ if } \pi > \omega_t,\]
\[\quad = 0 \text{ if } \pi < \omega_t.\]

This follows because while adults differ in their productivities their times spent in child care activities are perfect substitutes.

In view of (5) we can rewrite (3) and (4) as follows.

\[(3') \quad c_t + \int \omega_t(a) dF(a) \leq \int_{\omega_t}^{\infty} \pi dG_t(\pi).\]
\[(4') \quad \int T_t(a) dF(a) \leq C_t(\omega_t).\]

It's convenient to define the following.

\[(6) \quad Y(\pi, T, e, \pi') = Q(\pi, T, e) \text{ if } Q(\pi, T, e) > \pi',\]
\[= 0 \text{ otherwise.}\]

Using the above we can write total output in period \(t+1\) as follows.

\[(7) \quad \int_{\omega_{t+1}}^{\infty} \pi dG_{t+1}(\pi) = \int Y(\pi, T_t(a), c_t(a), \omega_{t+1}) dF(a).\]

Let \(\{p_t\}_{t=0}^{\infty}\) be a sequence of "efficiency prices" with \(p_t > 0\) for all \(t\).
Then any allocation which maximizes \( \Sigma_{t \geq 0} p_t C_t \) is efficient. We can interpret \( p_t / p_{t+1} = 1+r_t \) as the gross interest rate from \( t \) to \( t+1 \). We will primarily be interested in steady states. Without loss of generality we can look at the problem of maximizing \( (p_t C_t + p_{t+1} C_{t+1}) \) with respect to \( w_t \), \( T_t(a) \) and \( e_t(a) \) and then look at the steady state versions of the FONC's characterizing the solution.

Therefore, the problem is to maximize

\[
(8) \quad p_t \int_{w_t}^{\infty} \pi dG_t(\pi) - \int e_t(a) dF(a) + \int_{T_t(a), e_t(a), w_{t+1}}^{\infty} dT, \quad \text{subject to } (4').
\]

Let \( \lambda_t \) be the non-negative multiplier associated with the constraint (4'). The FONC's for the above problem are as follows.

\[
(9a) \quad -p_t \lambda_t G_t'(w_t) + \lambda_t C_t'(w_t) = 0, \\
(9b) \quad -p_t + p_{t+1} \delta Y(a, T_t(a), e_t(a), w_{t+1}) / \delta w \leq 0, \quad \text{with } = \text{if } e_t(a) > 0, \\
(9c) \quad -\lambda_t + p_{t+1} \delta Y(a, T_t(a), e_t(a), w_{t+1}) / \delta T \leq 0, \quad \text{with } = \text{if } T_t(a) > 0.
\]

The above FONC's can be simplified as follows.

\[
(10a) \quad \delta Y(a, T_t(a), e_t(a), w_{t+1}) / \delta w \leq 1+r_t, \quad \text{with } = \text{if } e_t(a) > 0, \\
(10b) \quad \delta Y(a, T_t(a), e_t(a), w_{t+1}) / \delta T \leq w_t(1+r_t), \quad \text{with } = \text{if } T_t(a) > 0.
\]

The steady state is characterized by the following equations.

\[
(11a) \quad \delta Y(a, T(a), e(a), w) / \delta e \leq 1+r, \quad \text{with } = \text{if } e(a) > 0, \\
(11b) \quad \delta Y(a, T(a), e(a), w) / \delta T \leq w(1+r), \quad \text{with } = \text{if } T(a) > 0, \\
(11c) \quad G(\pi) = m(a:Q(a,T(a),e(a)) \leq \pi), \\
(11d) \quad fT(a)dF(a) = G(w).
\]

Note that (11a,b) are the steady state versions of (10a,b), respectively, and that (11c) and (11d) are the steady state versions of (2) and (4'), respectively.

We could think of solving the above equations (for a given interest rate \( r \)) in the following way. Start with a guess for \( w \) and use equations (11a,b) to solve for the schedules \( T(a) \) and \( e(a) \). Then use (11c) to find \( G \) and (11d) to verify the guess for \( w \).

The solution has the following feature. As we already noted some adults
spend all their time producing and the rest of the adults spend all of their
time taking care of children. Some children will have positive amounts of
adult time and goods invested in them and will (when they become adults)
spend all of their time in production. The rest of the children will have
zero adult time and goods invested in them and will (when they become
adults) spend all of their time taking care of the next generation of
children.

Basically, the above conclusion is a result of the assumption that
while adults may differ in their productivities for producing goods they are
identical in terms of their productivities in taking care of children.

We now turn to various market arrangements and the resulting market
allocations.

Market arrangements

We have to say something about preferences first since children have no
resources with which to pay for adult time or investment in themselves and
they cannot borrow the needed resources since the adults (who can work and
produce and devote time to children) will not be around a period later to
collect on the loans and, therefore, will not lend to them. We will assume
that adults are matched one-to-one with children and that each adult cares
about her child altruistically. We will assume that bequests to children can
be either positive or negative, i.e., debts can be passed on as well. The
ability level of a future child is unknown currently and is randomly drawn
from the distribution F. We will only consider steady states.

With the above additional structure we can show that the following
market arrangement can reproduce the above efficient allocation.
(1) There are spot markets in child care at a wage of w in period t,
(2) There are one-period ahead complete insurance markets so that an adult
can insure against the ability level of one's grandchild next period. Let
\( q(a') \) denote the price of a claim which delivers a unit of consumption next
period if the ability level is \( a' \) and nothing otherwise.

The dynamic optimization problem of an adult can now be written as
follows. Let \( \beta \) be the discount factor applied by an adult to the future
welfare of her child, let \( U(c) \) be the current utility of an adult as a
function of consumption \( c \), and let \( V(\pi, a, b) \) be the value function of an
adult with productivity \( \pi \), having a child with ability \( a \), and a bequest
received (if positive) or debt due (if negative) of \( b \). This function solves
the following Bellman equation.
\[ V(\pi,a,b) = \max(U(c) + \beta J V(\pi',a',b(a'))dF(a')) \]

subject to:
\[ c = \max[\pi, w] - \omega T - e + b - J q(a')b(a')da'. \]
\[ \pi' = Q(a,T,e). \]

Note that if an adult's productivity exceeds \( w \) then this adult will devote her entire time to production and purchase child care at the wage \( w \). If the adult's productivity is below \( w \) then this adult can be thought of as providing child care to her own child in the amount \( T \) and using the remaining time \( (1-T) \) to provide child care for other children and earning \( w(1-T) \) by doing so. The first two terms on the right of (12b) embody this description.

Let \( \lambda \) be the non-negative multiplier associated with the budget constraint (12b). The FONCs and the envelope conditions for the above problem are as follows.

FONCs:
\[ U'(c) - \lambda = 0, \]
\[ -\lambda + \beta f(\partial Q/\partial e)(\delta V(\pi',a',b(a'))/\delta \pi)dF(a') = 0, \text{ with } e > 0, \]
\[ -\lambda \pi + \beta f(\partial Q/\partial T)(\delta V(\pi',a',b(a'))/\delta \pi)dF(a') = 0, \text{ with } T > 0, \]
\[ -\lambda q(a') + \beta (\delta V(\pi',a',b(a'))/\delta b)f'(a') = 0. \]

Envelope:
\[ \delta V(\pi,a,b)/\delta \pi = \lambda, \text{ if } \pi > w, \]
\[ = 0, \text{ if } \pi < w, \]
\[ \delta V(\pi,a,b)/\delta b = \lambda. \]

The above conditions are equivalent to the efficiency conditions (11) provided we set \( 1+r = 1/\beta \). Given complete insurance markets \( \lambda \) will be constant across states and will also be constant over time in a steady state. From (13d) and (14b) we then have that the contingent claims prices coincide with the probabilities and provide actuarially fair insurance. That is,

\[ q(a') = \beta F'(a'). \]

Obviously, the risk-free interest rate is given by
(16) \((1+r)^{-1} = \int q(a') da' = \int \beta F'(a') da' = \beta.\)

Further, if \(\pi' = Q(a, T, e) > w\), then using (14a) and the fact that \(\lambda\) is constant we can see that (13b, c) reduce to: \(\delta Q/\delta e = 1/\beta\) and \(\delta Q/\delta T = w/\beta\), respectively, which coincide with (11a, b) because in that case \(Y(a, T, e) = Q(a, T, e)\). If \(\pi' = Q(a, 0, 0) < w\), then (13b, c) imply that \(e = T = 0\). This coincides with (11a, b) because in that case \(Y(a, T, e, w) = 0\). Lastly, (11d) may be thought of as the market clearing condition for labor demand and labor supply in the child care market which determine the child care wage \(w\).

With the above market arrangement in mind it is now possible to discuss the possible sources of inefficiencies in a decentralised system.

**Sources of inefficiencies**

We can list below the various possible sources of inefficiencies.

1. Non-negativity of bequests
2. No insurance markets for insuring against future child ability
3. No labor market for child care

The non-negativity of bequests rules out a credit market so that adults with low productivity and high ability children are unable to borrow in order to undertake the efficient amount of investment in their children. Public education might mitigate this inefficiency somewhat. For instance, if a child's ability is currently not known or if the productivity of investment is independent of the child's ability level then efficiency dictates a uniform level of investment in all children regardless of ability provided they meet a minimum ability criterion. Borrowing constrained adults may undertake lower investments.

Lack of insurance markets will again result in inefficient levels of investment though it's not clear if there will be under or over investment.

A non-existent (or badly functioning) labor market in child care might force highly productive adults to devote time to children instead of production.

**Matching and Investment in children**

The above unisex model needs to be amended to take into account gender differences and the formation (or lack thereof) of two adult (presumably of opposite sexes) households with children. Imagine for now that at the beginning of each period there are single males and females with two
children. Suppose that adult males and females of low productivity are more likely to remain single than be matched with an adult of the opposite sex. Then such females who end up being unmatched have low resources and less time per child. Consequently, there is likely to be underinvestment in the children of such females who next period will have low productivity and be more likely than average to remain single and thereby continue the cycle of poverty.

Some further thoughts

The model as outlined up to now could easily be modified in the following ways.

(1) It is not necessary to assume a 1-1 matching between adults and children. One could assume that some adults are unmatched with children and other adults are matched with more than one child. Further, this could be made random so that future adults are randomly matched with a random number of children.

(2) Instead of assuming that preferences are altruistic one could consider other preferences. An example is one where an adult cares about her own consumption and the productivity level(s) of her child(ren). This formulation might be more convenient when we want to bring in matching considerations among adults. One could also include a direct utility benefit for the adult from a bequest left for children in order to permit some intertemporal trading.

(3) The key difficulty in analyzing matching among adults seems to be in generating positive sorting with respect to productivities. This also seems to depend on things like whether there is a market in child care or not etc.