W.T. Tutte showed that if $G$ is an arc transitive connected cubic graph then the automorphism group of $G$ is in fact regular on $s$-arcs for some $s \leq 5$. We analyze these arc transitive cubic graphs using the unifying concepts of the infinite cubic tree, $T_3$, and coverings. We are able to answer a large number of questions, open and otherwise. As an example, suppose $G$ is a 4-arc transitive cubic graph and the automorphism group of $G$ contains a 1-regular subgroup, then $G$ is a covering of Heawood's graph.
Abstract

W.T. Tutte showed that if $G$ is an arc transitive connected cubic graph then the automorphism group of $G$ is in fact regular on $s$-arcs for some $s \leq 5$. We analyze these arc transitive cubic graphs using the unifying concepts of the infinite cubic tree, $I_3$, and coverings. We are able to answer a large number of questions, open and otherwise. As an example, suppose $G$ is a 4-arc transitive cubic graph and the automorphism group of $G$ contains a 1-regular subgroup, then $G$ is a covering of Heawood's graph.
Regular Groups of Automorphisms of Cubic Graphs

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0. Introduction.

This paper is concerned with s-regular groups of automorphisms of connected cubic graphs. (See next section for definitions.) The subject was started by W.T. Tutte [13] by proving the basic fact that \( s \leq 5 \) in the case of finite cubic graphs. (See also [15].) He also proved in that case that if the graph is 1-transitive then it must be s-regular for some \( s, 1 \leq s \leq 5 \). With slight modifications these results remain valid for infinite cubic graphs.

Let \( G \) be a connected cubic graph and let \( \{u,v\} \) be an edge of \( G \). Let \( A \) be an s-regular subgroup of \( \text{Aut}(G) \). Then we associate to \( A \) the amalgam \( (A(u), A[u,v]) \) where \( A(u) \) is the fixer of \( u \) in \( A \) and \( A[u,v] \) is the stabilizer of \( \{u,v\} \) in \( A \). This amalgam is independent (up to isomorphism) of the chosen edge \( \{u,v\} \). Moreover, if \( s = 1,3, \) or \( 5 \) this amalgam is even independent of the choice of \( G \) and \( A \) and if \( s = 2 \) or \( 4 \) there are two possible amalgams in each case. Hence we obtain seven possible types of s-regular groups (1 ≤ s ≤ 5): 1', 2', 2", 3', 4', 4", and 5'. The group of type \( s' \) is s-regular and has an involution which flips an edge. The groups of type \( s" \) are s-regular and they do not possess any involution flipping an edge.

Let \( r_3 \) be the cubic tree. We show that \( \text{Aut}(r_3) \) contains regular subgroups of all seven types, moreover two regular subgroups of the same type are conjugate.
in \( \text{Aut}(\Gamma_3) \). The possible inclusions among regular subgroups of \( \text{Aut}(\Gamma_3) \) are determined. For instance, an \( s \)-regular group with \( s = 2 \) or \( 3 \) is not contained in any \( t \)-regular subgroup with \( t = 4 \) or \( 5 \). Also a \( 1 \)-regular subgroup is contained in precisely one subgroup of type \( 2' \), one subgroup of type \( 3' \), two subgroups of type \( 4' \), two subgroups of type \( 5' \) and it is not contained in any subgroup of type \( 2'' \) or \( 4'' \). Some of these assertions are valid in arbitrary cubic graphs. We determine the centralizers and normalizers in \( \text{Aut}(\Gamma_3) \) of \( s \)-regular subgroups (\( 1 \leq s \leq 5 \)). It is easy to see that a subgroup of type \( 1' \) of \( \text{Aut}(\Gamma_3) \) is isomorphic to the modular group \( C_2 \ast C_3 \) (the free product of a cyclic group of order 2 and a cyclic group of order 3).

It was shown by the first author in [5] that if \( A_5' \) is a subgroup of type \( 5' \) of \( \text{Aut}(\Gamma_3) \) then the pair \( (\Gamma_3, A_5') \) has a certain universal property. Namely, if \( (G, A) \) is another pair consisting of a connected cubic graph \( G \) and a subgroup \( A \) of type \( 5' \) of \( \text{Aut}(G) \) then there is a covering map \( \Gamma_3 \to G \) which is compatible with the actions of \( A_5' \) on \( \Gamma_3 \) and \( A \) on \( G \). This gives rise to a homomorphism \( f: A_5' \to A \) which is onto and one can reconstruct the pair \( (G, A) \) from the pair \( (\Gamma_3, A_5') \) and the knowledge of the subgroup \( \ker f \) of \( A_5' \).

Now, these results have been proved in detail in this paper for all seven types of regular groups. The proofs are somewhat simpler than those used in [5].

It turns out that to every normal subgroup \( N \) of a regular group \( A \) of a fixed type of \( \text{Aut}(\Gamma_3) \) (with finitely many exceptions) one can canonically associate a pair \( (G_N, A_N) \) consisting of a connected cubic graph \( G_N \) and a regular subgroup \( A_N \) of the same type of \( \text{Aut}(G_N) \).
Moreover, in this way we obtain all such pairs \((G,B)\) (up to isomorphism). This correspondence \(N \mapsto (G_N, A_N)\) is either one-one or two-one, i.e., the same pair can be obtained from at most two normal subgroups. Hence the problem of classifying pairs \((G,B)\) of the given type reduces to the problem of classifying normal subgroups of \(A\).

This latter problem is still far from its solution. For instance, if we consider the type \(1'\) then \(A\) is the modular group and in spite of an enormous number of papers on normal subgroups of it (see the references, in [9]) the final answer is still lacking. It is even unknown what are the simple quotients of the modular group, i.e., which simple groups can be generated by one element of order 2 and one element of order 3.

These are the main ideas of this paper but an interested reader will find in it many other results which have not been mentioned above. As a sample, we now mention the following result: If \(G\) is a connected cubic graph and \(\text{Aut}(G)\) contains both a 2-regular and a 4-regular subgroup then \(G\) is a cubic tree.
1. Terminology and notation.

For any set $V$ let $P_2(V)$ be the set of 2-element subsets of $V$. A graph (more precisely, a combinatorial graph) is an ordered pair $G = (V,E)$ where $V$ is a set and $E \subseteq P_2(V)$. The elements of $V$ (resp., $E$) are vertices (resp., edges) of $G$. When $\{a,b\} \in E$ then $a \neq b$ and we say that $a$ and $b$ are adjacent (or that they are neighbours) and that the edge $\{a,b\}$ is joining them. The set of all neighbours of a vertex $a$ will be denoted by $G(a)$. The valency of a vertex $a$ is the cardinality of the set $G(a)$. We say that $G$ is regular if all vertices of $G$ have the same valency. A cubic graph is a regular graph of valence 3.

The graph $\text{Path}(s)$, $s \geq 0$, is the graph $(V,E)$ where $V = \{0,1,\ldots,s\}$ and $E$ consists of the pairs $\{i,i+1\}$, $0 \leq i \leq s-1$.

The graph $\text{Path}(\infty)$ has vertices the non-negative integers and the edges $\{i,i+1\}$ for $i \geq 0$. The graph $\text{Path}(\infty,\infty)$ has vertices all integers and the edges $\{i,i+1\}$, $i \in \mathbb{Z}$ (the set of rational integers).

The graph $\text{Cir}(s)$, $s \geq 3$, is the graph obtained from $\text{Path}(s-1)$ by adding the edge $\{0,s-1\}$.

A homomorphism $f: G \to G'$ from a graph $G = (V,E)$ to a graph $G' = (V',E')$ is a map $f: V \to V'$ such that $\{a,b\} \in E$ implies that $\{f(a),f(b)\} \in E'$. Hence, we have that a homomorphism induces a map $E \to E'$. We say that a homomorphism $f: G \to G'$ is injective (resp., surjective) if both maps $f: V \to V'$ and the induced map $E \to E'$ are injective (resp., surjective). In fact, if $f: V \to V'$ is injective then also the induced map $E \to E'$ is injective. If a homomorphism $f: G \to G'$
is both injective and surjective then it has an inverse and we say that $f$ is an isomorphism. An automorphism of a graph $G$ is an isomorphism $G \to G$. Two graphs $G$ and $G'$ are isomorphic if there exists an isomorphism $G \to G'$.

A graph $G = (V,E)$ is a subgraph of the graph $G' = (V',E')$ if $V \subseteq V'$ and $E \subseteq E'$. In that case the inclusion map $V \to V'$ is a graph homomorphism. We say that a subgraph $G$ of $G'$ is full if $E = E' \cap P_2(V)$. If $f: G \to G'$ is a graph homomorphism then the image of $f$ is a subgraph of $G'$ whose vertices are $f(a)$, for $a \in V$ and whose edges are $\{f(a), f(b)\}$ for $\{a,b\} \in E$. We say that a graph homomorphism is full if its image is a full subgraph.

It is clear that $\text{Path}(s-1)$ is a subgraph of $\text{Cir}(s)$ but it is not full. If $G = (V,E)$ is a graph and $V'$ a subset of $V$, then there exists a unique full subgraph $G'$ of $G$ whose vertex set is $V'$. In fact $G' = (V',E')$ where $E' = E \cap P_2(V')$. We say that $G'$ is the subgraph of $G$ spanned by $V'$.

A path of length $s$ (or, an $s$-path) in a graph $G$ is a homomorphism $f: \text{Path}(s) \to G$. For such path we say that $f(0)$ is its origin and $f(s)$ its end and that it joins $f(0)$ to $f(s)$. An $\omega$-path (or, simply infinite path) in $G$ is a homomorphism $f: \text{Path}(\omega) \to G$. An $\omega$-path has only an origin $f(0)$ and no end. A $\omega,\omega$-path (or, doubly infinite path) in $G$ is a homomorphism $f: \text{Path}(\omega,\omega) \to G$. It has neither an origin nor an end. An $s$-path is closed if $f(0) = f(s)$, otherwise it is open.
If $f$ is an $s$-path in $G$ then its opposite $s$-path $f'$ is defined by $f'(i) = f(s-i)$, $0 \leq i \leq s$. Similarly, if $f$ is a doubly infinite path in $G$ then its opposite doubly infinite path $f'$ is defined by $f'(i) = f(-i)$ for $i \in \mathbb{Z}$.

If $a$ and $b$ are vertices of $G$ we let $\delta(a,b)$ be the length of a shortest path joining $a$ to $b$ if such path exists and we put $\delta(a,b) = \infty$ otherwise. Then $\delta$ is a distance function. Of course, $\delta(a,b) = 1$ iff $a$ and $b$ are neighbours. It is clear that the relation $\delta(a,b) < \infty$ between vertices $a,b$ of a graph $G$ is an equivalence relation. The subgraphs of $G$ spanned by the equivalence classes of vertices (for this relation) are called the connected components of $G$. If $G$ consists of only one component then we say that $G$ is connected.

A circuit of length $s$ (or, an $s$-circuit) in a graph $G$ is a subgraph of $G$ isomorphic to $\text{Cir}(s)$, $s \geq 3$. If a graph $G$ has no circuits then we say that $G$ is a forest. A connected forest is a tree. A regular tree of valency $n$ will be denoted by $\Gamma_n$. Many of our results are about the automorphisms of $\Gamma_3$.

A graph homomorphism $f; G \to G'$ is locally injective (resp., locally surjective) if for every vertex $a$ of $G$ the map $f_a: G(a) \to G'(f(a))$, obtained from $f$ by restricting its domain and codomain, is injective (resp., surjective). If $f$ is both locally injective and locally surjective then we shall say that it is a local isomorphism.

An $s$-arc in a graph $G$ is an $s$-path $f$ which is locally injective i.e., such that $f(i) \neq f(i+2)$ for $0 \leq i \leq s - 2$. An $\infty$-arc (simply infinite arc) or an $(-\infty, \infty)$-arc (doubly infinite arc) are defined analogously. A $0$-arc may be identified with its origin (=end). A $1$-arc is also called an
ory oriented edge. If \( f \) is an \( s \)-arc in \( G \) then the opposite \( s \)-path \( f' \) is also an \( s \)-arc. The same holds for doubly infinite arcs.

A graph homomorphism \( f: G \to G' \) is called a covering if both \( G \) and \( G' \) are connected and \( f \) is a local isomorphism. For instance, we have a covering \( f: \text{Path}(\infty,\infty) \to \text{Cir}(s) \) defined as follows: \( f(i) \) is the remainder of the division of \( i \) by \( s \). The restriction of \( f \) to its subgraph \( \text{Path}(\infty) \) is not a covering because the condition defining local isomorphism is not verified at the vertex \( 0 \). We shall give later more interesting examples of coverings. Every covering is surjective.

The automorphisms of a graph \( G = (V,E) \) form a group \( \text{Aut}(G) \). It is clear that every automorphism of \( G \) induces a permutation of both \( V \) and \( E \). If \( S \) is an \( s \)-arc and \( \alpha \in \text{Aut}(G) \) then \( \alpha \circ S \) is also an \( s \)-arc. In this way every \( \alpha \) induces a permutation of the set of \( s \)-arcs in \( G \). Similarly, every \( \alpha \in \text{Aut}(G) \) permutes the set of \( \infty \)-arcs and the set of \( (\infty,\infty) \)-arcs in \( G \).

A subgroup \( A \leq \text{Aut}(G) \) is \( s \)-transitive (resp., \( s \)-regular) if the action of \( A \) on the set of \( s \)-arcs of \( G \) is transitive (resp., regular, i.e., sharply transitive). \( A \) is \( \omega \)-transitive if it is \( s \)-transitive for all integers \( s \geq 0 \). The subgroup \( A \) is \( (\infty,\infty) \)-transitive (resp., \( (\infty,\infty) \)-regular) if its action on the set of doubly infinite arcs in \( G \) is transitive (resp., regular). A graph \( G \) is \( s \)-transitive (resp., \( s \)-regular) if \( \text{Aut}(G) \) is \( s \)-transitive (resp., \( s \)-regular).

If \( A \) is \( s \)-transitive then it is also \( t \)-transitive for \( 0 \leq t \leq s \). It is clear that \( 0 \)-transitive means vertex-transitive and that every \( 1 \)-transitive group is also edge-transitive. The converse of the last assertion is not valid. For instance, if \( G = K_{m,n} \) (the complete bipartite graph with \( m \) "white" and \( b \) "black" vertices) and \( A = \text{Aut}(G) \) with
m \neq n$ then $A$ is edge-transitive but not vertex-transitive.

Let $A$ be a group and $G = (V, E)$ a graph. Let $f: A \times V \to V$ be a map and let us write $\alpha \cdot a$ instead of $f(\alpha, a)$ for $\alpha \in A$ and $a \in V$. We say that $f$ is an action of $A$ on $G$ if the map $f_\alpha: V \to V$ defined by $f_\alpha(a) = \alpha \cdot a$ is an automorphism of $G$ for every $\alpha \in A$, $\alpha \cdot (\beta \cdot a) = (\alpha \beta) \cdot a$ for all $\alpha, \beta \in A$ and $a \in V$, and $1 \cdot a = a$ for all $a \in V$.

If $f$ is an action of $A$ on $G$ then the map $A \to \text{Aut}(G)$ defined by sending $\alpha$ to $f_\alpha$ is a group homomorphism. Conversely, given a group homomorphism $A \to \text{Aut}(G)$ one can use it to define an action of $A$ on $G$. In most of this paper $A$ will be a subgroup of $\text{Aut}(G)$ and hence the action of $A$ on $G$ is the one obtained from the inclusion map $A \to \text{Aut}(G)$.

We will consider the category whose objects are ordered pairs $(G, A)$ consisting of a graph $G$ and a group $A$ acting on it. The morphisms in this category are pairs $(f, g)$ of maps

$$(f, g): (G, A) \to (G', A')$$

where $f: G \to G'$ is a graph homomorphism, $g: A \to A'$ a group homomorphism, and they are compatible in the sense that the diagram

$$
\begin{array}{ccc}
G & \xrightarrow{\alpha} & \hat{G} \\
\downarrow f & & \downarrow f \\
G' & \xrightarrow{g(\alpha)} & G'
\end{array}
$$

is commutative, i.e., $g(\alpha) \circ f = f \circ \alpha$, for all $\alpha \in A$. 


A covering morphism is a morphism \((f, g): (G, A) \rightarrow (G', A')\) such that \(f\) is a graph covering and \(g\) is a surjective homomorphism.

We shall say that a graph covering \(f: G \rightarrow G'\) is compatible with the action of a group \(A\) on \(G\) if \(f(a) = f(b)\) implies that \(f(a \cdot a) = f(a \cdot b)\) for all \(a, b \in V\) and \(a \in A\). If this is so then we can define an action of \(A\) on \(G'\) as follows: if \(x \in V'\) choose \(a \in V\) such that \(f(a) = x\) and define \(\alpha \cdot x = f(\alpha \cdot a)\) for all \(\alpha \in A\). We shall say that this action of \(A\) on \(G'\) is induced by the action of \(A\) on \(G\).

A shunting in a graph \(G\) is an ordered pair \((a, \alpha)\) where \(a\) is a vertex of \(G\) and \(\alpha\) an automorphism of \(G\) such that \(\alpha(a)\) is adjacent to \(a\) and \(\alpha^2(a) \neq a\). Every shunting \((a, \alpha)\) determines a doubly infinite arc \(S\) in \(G\) by defining \(S(i) = \alpha^i(a), i \in \mathbb{Z}\). The image of \(S\) in \(G\) will be called the trajectory of \((a, \alpha)\). It is easy to see that it is isomorphic to either \(\text{Path}(\infty, \infty)\) or to \(\text{Cir}(s)\) for some \(s \geq 3\).

Let \(A \leq \text{Aut}(G)\). Then a shunting \((a, \alpha)\) in \(G\) is an \(A\)-shunting if \(\alpha \in A\). The group \(\text{Aut}(G)\) acts on the set of all shuntings in \(G\) by defining

\[ \beta \cdot (a, \alpha) = (\beta(a), \beta \alpha \beta^{-1}) \]

for \(\beta \in \text{Aut}(G)\) and any shunting \((a, \alpha)\). We say that two shuntings \((a, \alpha)\) and \((b, \beta)\) are \(A\)-conjugate (or conjugate in \(A\)) if there exists \(\gamma \in A\) such that \(\gamma \cdot (a, \alpha) = (b, \beta)\). If \(A = \text{Aut}(G)\) then we shall say conjugate instead of \(A\)-conjugate. For instance, if \((a, \alpha)\) is a shunting then also \((\alpha^k(a), \alpha)\) is a shunting. They are conjugate because
\( a^k \cdot (a, \alpha) = (a^k(a), \alpha) \) and they have the same trajectory.

If \((a, \alpha)\) is a shunting so is \((a, \alpha^{-1})\) and we say that they are **opposite** to each other. We say that two shuntings \((a, \alpha)\) and \((b, \beta)\) have overlaps \((s \geq 1)\) if \(a = b, \alpha^k(a) = \beta^k(a)\) for \(0 \leq k \leq s\) and \(\alpha^{-1}(a) \neq \beta^{-1}(a), \alpha^{s+1}(a) \neq \beta^{s+1}(a)\).

A graph \(G = (V, E)\) is **finite** if the set \(V\) is finite. Many results in the literature are stated in the framework of finite graphs although they are valid without that restriction. We shall use these results whenever it is obvious that the finiteness restriction can be removed.

A semidirect product \(S\) of two groups \(A\) and \(B\) will be denoted by \(A \rtimes B = S\); \(A\) is a normal subgroup of \(S\) and \(B\) acts on \(A\). Our group-theoretical notation is standard; we just mention that \(S_n\) is the symmetric group of degree \(n\), \(D_n\) the dihedral group of order \(2n\), and \(C_n\) is the cyclic group of order \(n\).
2. Vertex-fixers and the Even Subgroup.

Let $G = (V,E)$ be a graph and $A \leq Aut(G)$. The fixer in $A$ of a vertex $v \in V$ is the subgroup $A(v)$ of $A$ consisting of all $\alpha \in A$ such that $\alpha(v) = v$. If $v_1, \ldots, v_k$ are vertices of $G$ we put

$$A(v_1, \ldots, v_k) = \bigcap_{i=1}^k A(v_i).$$

The even subgroup $A^+$ of $A$ is the subgroup generated by all $A(v), \ v \in V$.

**Proposition 1.** Let $G$ be connected, $A$ 1-transitive and let $\{x,y\} \in E$. Choose $\xi \in A$ such that $\xi(x) = y$. Then

1. $A^+$ is generated by $A(x)$ and $A(y)$;
2. $A$ is generated by $A(x)$ and $\xi$;
3. $(A:A^+) = 1$ or 2;
4. $(A:A^+) = 2$ iff $G$ is bipartite.

**Proof.** Let $B$ be the subgroup of $A^+$ generated by $A(x)$ and $A(y)$. We claim that for every $v \in V$ there exists $\alpha \in B$ such that $\alpha(v) \in \{x,y\}$. The proof is by induction on the distance $\delta$ from $v$ to $\{x,y\}$. If $\delta = 0$ then $v \in \{x,y\}$ and we can take $\alpha = 1$. Assume that $\delta \geq 1$ and, say, $\delta(v,x) = \delta$, $\delta(v,y) = \delta + 1$. There is a vertex $z \in G(a)$ such that $\delta(v,z) = \delta - 1$. Since $A$ is $1$-transitive, there exists $\beta \in A(x)$ such that $\beta(z) = y$. Then $\delta(\beta(v),y) \leq \delta - 1$ and by induction hypothesis there exists $\gamma \in B$ such that $\gamma(\beta(v)) \in \{x,y\}$. Thus we can take $\alpha = \gamma \beta$ and our claim is proved.
Let \( v \in V \). By what we proved, there is an \( \alpha \in B \) such that 
\( \alpha(v) = x \) or \( y \). Then \( \alpha A(v) \alpha^{-1} \subseteq A(x) \) or \( A(y) \) and so \( A(v) \subseteq B \).

Hence \( A^+ \subseteq B \) and since also \( B \subseteq A^+ \), we have \( A^+ = B \). Thus (i) is proved.

For \( v \in V \) and \( \alpha \in A \) we have \( \alpha A(v) \alpha^{-1} = A(\alpha(v)) \), which implies that \( A^+ \) is a normal subgroup of \( A \).

Let \( \alpha \in A \). There is a \( \beta \in A^+ \) such that \( \beta(\alpha(x)) = x \) or \( y \).

If \( \beta(\alpha(x)) = x \) then \( \beta \alpha \in A(x) \), \( \alpha \in A^+ \). If \( \beta(\alpha(x)) = y \) then 
\( \xi^{-1} \beta \alpha \in A(x) \), \( \alpha \in \beta^{-1} \xi A(x) \subseteq A^+ \xi A^+ = \xi A^+ \). Hence \( A \subseteq A^+ u \xi A^+ \) and (iii) is proved.

Let \( C \) be the subgroup of \( A \) generated by \( A(x) \) and \( \xi \). Then \( \xi(x) = y \) implies that \( \xi A(x) \xi^{-1} = A(y) \) and hence \( A(2) \subseteq C \). Therefore
\( A^+ \subseteq C \). Since also \( \xi \in C \) we have \( A = A^+ u \xi A^+ \subseteq C \). Hence we have \( C = A \) and (ii) is proved.

Assume that \( G \) is bipartite. Then there is a partition 
\( V = V_1 u V_2 \) such that every edge of \( G \) has one vertex in \( V_1 \), and one in \( V_2 \). For each \( v \in V \) it is easy to see that if \( \alpha \in A(v) \) then 
\( \alpha(V_i) = V_i, \ i = 1,2 \). Since \( A^+ \) is generated by \( A(v) \), \( v \in V \) it follows that for every \( \alpha \in A^+ \) we have \( \alpha(V_i) = V_i, \ i = 1,2 \). But \( \xi(x) = y \) and hence \( \xi \in A^+ \). Hence, if \( G \) is bipartite then \( (A:A^+) = 2 \).

Assume now that \( G \) is not bipartite. Since \( A \) is 1-transitive, there exists an odd circuit whose consecutive vertices are

\[ v_0 = x, v_1 = y, v_2, \ldots, v_{n-1}, v_n = v_0 = x. \]

Thus \( n = 2m + 1 \) is odd. There exist \( \alpha_i \in A(v_{2i+1}) \) \( 0 \leq i \leq m \), such that \( \alpha_i(v_{2i}) = v_{2i+2} \) for \( 0 \leq i \leq m - 1 \) and \( \alpha_m(v_{2m}) = y \). Then we
can take $\xi = \alpha_m \alpha_{m-1} \cdots \alpha_1 \alpha_0$ because $\xi(x) = y$ and since $\xi \in A^+$ we have by (ii) that $A = A^+$.

The proof of (iv) is completed.

The part (ii) is identical to Lemma 2.3 of R.C. Miller [8].
3. Canonical Involutions.

Let $G = (V,E)$ be a connected, cubic graph and $A$ an $s$-regular group of automorphisms of $G$, $s$ being a positive integer. It is well-known [13] that we must have $s \leq 5$ and that the girth of $G$ is $\geq 2s - 2$. The $s$-regularity of $A$ implies that $|A(x)| = 3 \cdot 2^{s-1}$ for every $x \in V$. Similarly, if $\{x,y\} \in E$ then $|A(x,y)| = 2^{s-1}$. If also $\{y,z\} \in E$, $x \neq z$ and $s \geq 2$ then $|A(x,y,z)| = 2^{s-2}$, etc.

When $s = 2$ or $4$ we shall associate to each edge $a = \{x,y\}$ of $G$ an involution $\tilde{a}$ in $A(x,y)$. When $s = 3$ or $5$ we shall associate to each vertex $a \in V$ an involution $\tilde{a} \in A(a)$.

Proposition 2. Let $s = 2$ and let $a,b,c,d$ be edges and $x,y,z,u,v$ vertices as shown on Fig. 1. Then

(i) there is a unique non-identity element $\tilde{a}$ in $A$ such that $\tilde{a}(x) = x$, $\tilde{a}(y) = y$;

(ii) $\tilde{a}$ is an involution;

(iii) $\tilde{a}$ moves every vertex at distance 1 from $a$;

(iv) if $a \in A$ and $a(a) = e$ then $\tilde{a}a^{-1} = \tilde{e}$;

(v) $(\tilde{a},\tilde{b})^3 = 1$, $\tilde{aba} = \tilde{d}$;

(vi) $A(y) = \langle \tilde{a},\tilde{b} \rangle \cong D_3$.

![Fig. 1](image-url)
Proof. (i) and (ii) follow from \(|A(x,y)| = 2\). (iii) is immediate from 2-regularity of \(A\). (iv) follows from (i) and the fact that \(\tilde{a}a^{-1}\) fixes the vertices \(a(x)\) and \(a(y)\) of \(e\).

(v) \(\tilde{a}(z) = v\) by (iii) and \(\tilde{a}(y) = y\). By (iv), \(\tilde{a}(b) = d\) implies \(\tilde{a}b\tilde{a} = d\). Similarly, \(\tilde{b}(a) = d\) implies \(\tilde{b}a\tilde{b} = d\). Hence \((\tilde{a}b)^3 = (\tilde{a}b)(\tilde{a}b) = \tilde{d} = 1\).

(vi) We have \(A(y) = \langle \tilde{a}, \tilde{b} \rangle\) because \(|A(y)| = 6\); \(\tilde{a}, \tilde{b} \in A(y)\) and \(\tilde{a} \neq \tilde{b}\). Indeed, \(\tilde{a}(z) = v\) and \(\tilde{b}(z) = z\) show that \(\tilde{a} \neq \tilde{b}\). Since \(\tilde{ab}\) has order 3 by (v) we must have \(\langle \tilde{a}, \tilde{b} \rangle = D_3\).

Proposition 3. Let \(s = 3\) and let \(a, b, c, d, e\) be vertices of \(G\) as in Fig. 2. Then

(i) there is a unique non-identity element \(\tilde{b}\) in \(A\) which fixes \(\tilde{b}\) and all its neighbours;

(ii) if \(a \in A\) and \(a(b) = x\) then \(a\tilde{b}a^{-1} = \tilde{x}\);

(iii) \(\tilde{b}\) is an involution and belongs to the center of \(A(b)\);

(iv) \(\tilde{b}\) moves every vertex at distance 2 from \(b\);

(v) \(\tilde{a}\tilde{b} = \tilde{b}\tilde{a}\) and \((\tilde{a}\tilde{c})^3 = 1\);

(vi) \(A(a, b) = \langle \tilde{a}, \tilde{b} \rangle\) is a four-group and \(A(b) = \langle \tilde{a}, \tilde{b}, \tilde{c} \rangle = \langle \tilde{a}, \tilde{c} \rangle \times \langle \tilde{b} \rangle \cong D_3 \times C_2 \cong D_6\).

Fig. 2
Proof. (i) and the first part of (iii) follow from $A(a,b,c) = A(a,b,c,e)$ and $|A(a,b,c)| = 2$. (iv) follows from 3-regularity of $A$. (ii) follows from (i) since $\alpha b \alpha^{-1}$ fixes $x$ and all its neighbours. The second assertion of (iii) follows from (ii) by taking $\alpha \in A(b)$.

(v) $\tilde{a}b = \tilde{b}a$ by the second assertion of (iii) because $\tilde{a} \in A(b)$. By (iv), $\tilde{a}(c) = e$. Hence, by (ii), $\tilde{a}c\tilde{a} = \tilde{e}$. Similarly, $\tilde{c}(a) = e$ and $\tilde{c}c\tilde{a}c = \tilde{e}$. Thus $(\tilde{a}c)^3 = (\tilde{a}c)(c\tilde{a}c) = \tilde{ee} = 1$.

(vi) We have $\tilde{a} \neq \tilde{b}$ because $\tilde{a}(c) = e$ and $\tilde{b}(c) = c$. Hence $<\tilde{a}, \tilde{b}>$ is a four-group, using (v), and since $<\tilde{a}, \tilde{b}> \leq A(a,b)$ and $|A(a,b)| = 4$ we have $A(a,b) = <\tilde{a}, \tilde{b}>$.

Since $\tilde{c}(a) = e$ we have $\tilde{c} \notin A(a,b)$. Then we must have $A(b) = <\tilde{a}, \tilde{b}, \tilde{c}>$ since $|A(b)| = 12$. It follows from (v) that $<\tilde{a}, \tilde{c}> \cong D_3$ and the 3 involutions in $<\tilde{a}, \tilde{c}>$ are $\tilde{a}, \tilde{c}$ and $\tilde{e} = \tilde{a}c\tilde{a} = \tilde{c}c\tilde{a}$. Hence $\tilde{b} \notin <\tilde{a}, \tilde{c}>$ and by using (iii) we have $A(b) = <\tilde{a}, \tilde{c}> \times <\tilde{b}>$.

Proposition 4. Let $s = 4$ and let $a, b, c, d, e, f$ be the edges and $v_0, v_1, \ldots, v_7$ the vertices as indicated on Fig 3. Then

(i) There is a unique non-identity element $\tilde{b}$ in $A$ which fixes all vertices at distance $\leq 1$ from $b$;

(ii) $\tilde{b}$ is an involution;

(iii) $\tilde{b}$ moves every vertex at distance 2 from $b$;

(iv) if $a \in A$ and $a(b) = x$ then $a\tilde{b}a^{-1} = x$;

(v) $\tilde{a}\tilde{b} = \tilde{b}\tilde{a}$, $(\tilde{a}\tilde{c})^2 = \tilde{b}$, $(\tilde{a}\tilde{d})^3 = 1$;

(vi) $A(v_0, v_1, v_2) = <\tilde{a}, \tilde{b}>$ is a four-group, $A(v_1, v_2) = <\tilde{a}, \tilde{b}, \tilde{c}> \cong D_4$, $A(v_2) = <\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}> \cong S_4$. 
Proof. (i) and (ii) follow from $|A(v_0, v_1, v_2, v_3)| = 2$. (iii) follows from 4-regularity of $A$. (iv) follows from (i) and the fact that $\alpha \beta \alpha^{-1}$ fixes every vertex at distance $\leq 1$ from $x$.

(v) $\tilde{a} \beta = \beta \tilde{a}$ follows from (iv) since $\tilde{a}(b) = b$. By (i) and (iii) we have $\tilde{a}(c) = f$. Hence, by (iv), $\tilde{a}c\tilde{a} = f$. Since $<\tilde{b}, \tilde{c}>$ is a four-group we must have $f = \tilde{b}c$. Therefore $\tilde{a}c\tilde{a} = \tilde{b}c$, i.e., $(\tilde{ac})^2 = \tilde{b}$.

Since $(\tilde{d}\tilde{a})(v_3) = \tilde{d}(v_5) = v_1$ and $(\tilde{d}\tilde{a})(v_2) = v_2$ we must have $(\tilde{d}\tilde{a})(v_4) = v_0$ or $v_7$. In both cases this vertex is fixed by $\tilde{a}$, i.e., $\tilde{a}\tilde{d}\tilde{a}(v_4) = \tilde{d}\tilde{a}(v_4)$. Thus $\tilde{a}\tilde{d}\tilde{a}(v_4) = \tilde{d}\tilde{a}(v_4)$ since $\tilde{d}(v_4) = v_4$ and consequently $(\tilde{a}\tilde{d})^3(v_4) = v_4$. Similarly, we have $(\tilde{a}\tilde{d})^3(v_0) = v_0$. Hence $(\tilde{a}\tilde{d})^3$ fixes the vertices $v_i$, $0 \leq i \leq 4$ and we have $(\tilde{a}\tilde{d})^3 = 1$ by 4-regularity of $A$.

(vi) $A(v_0, v_1, v_2) = <\tilde{a}, \tilde{b}>$ because $|A(v_0, v_1, v_2)| = 4$. Since $\tilde{c}$ moves $v_0$ we have $A(v_1, v_2) = <\tilde{a}, \tilde{b}, \tilde{c}>$. Since $(\tilde{ac})^2 = \tilde{b}$, by (v), we have $<\tilde{a}, \tilde{b}, \tilde{c}> = <\tilde{a}, \tilde{c}> \cong D_4$. Since $\tilde{d}$ moves $v_1$ we have $\tilde{d} \neq <\tilde{a}, \tilde{b}, \tilde{c}> = A(v_1, v_2)$.

Fig. 3
Using that \(|A(v_2)| = 24\) we conclude that \(A(v_2) = \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \rangle\). Using (v) we see that \(\langle \tilde{b}, \tilde{c} \rangle\) is a normal subgroup of \(A(v_2)\). We also have \(\langle \tilde{a}, \tilde{d} \rangle \cong D_3\) and it follows [7; p. 201, Hilfssatz 8.17] that \(A(v_2) \cong S_4\).

**Proposition 5.** Let \(s = 5\) and let \(a, b, \ldots\) be the vertices of \(G\) as indicated on Fig. 4. Then

1. there is a unique non-identity element \(\tilde{c}\) in \(A\) which fixes all vertices at distance \(\leq 2\) from \(c\);

1. \(\tilde{c}\) is an involution and belongs to the center of \(A(c)\);

1. if \(a \in A\) and \(a(c) = x\) then \(a\tilde{c}a^{-1} = \tilde{x}\);

1. \(\tilde{c}\) moves every vertex at distance 3 from \(c\);

1. \(\tilde{a}\tilde{b} = \tilde{b}\tilde{a}, \quad \tilde{a}\tilde{c} = \tilde{c}\tilde{a}, \quad (\tilde{a}\tilde{d})^2 = \tilde{b}\tilde{c}, \quad (\tilde{a}\tilde{e})^3 = 1\);

1. \(A(a, b, c, d) = \langle \tilde{b}, \tilde{c} \rangle\) is a four-group, \(A(b, c, d) = \langle \tilde{b}, \tilde{c}, \tilde{d} \rangle\) is elementary abelian group of order 8, \(A(b, c) = \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \rangle = \langle \tilde{a}, \tilde{d} \rangle \times \langle \tilde{b} \rangle = \langle \tilde{a}, \tilde{d} \rangle \times \langle \tilde{c} \rangle \cong D_4 \times C_2\), \(A(c) = \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e} \rangle = \langle \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e} \rangle \times \langle \tilde{c} \rangle \cong S_4 \times C_2\).

![Fig. 4](image)

**Proof.** See Lemmas 1 - 3 and Theorem 1 of [5].
4. Edge-stabilizers.

Let $G$ and $A$ be as in the previous section. If $\{x,y\} \in E$ then we denote by $A[x,y]$ the stabilizer in $A$ of that edge, i.e., the subgroup of $A$ consisting of all $\alpha \in A$ such that $\{\alpha(x),\alpha(y)\} = \{x,y\}$. It is obvious that $A(x,y)$ is a subgroup of $A[x,y]$ of index 2 because $A$ is 1-transitive.

The groups $A(x,y)$ have been determined in the previous section. Now we shall find all possible groups $A[x,y]$ for $2 \leq s \leq 5$.

**Proposition 6.** Let $s = 2$ and let $t \in A$ be defined by $t(x) = u$, $t(y) = z$, $t(z) = y$ where we use notation from Fig. 1. Then $A[y,z] = \langle b, t \rangle$ and either

(i) $t^2 = 1$ and $A[y,z]$ is a four-group; or

(ii) $t^2 = b$ and $A[y,z] = \langle t \rangle \cong C_4$.

**Proof.** Since $t \in A(y,z)$ we have either $t^2 = 1$ or $t^2 = b$. Since $t(b) = c$ we have $tbt^{-1} = b$, i.e., $t$ and $b$ commute. If $t^2 = 1$ we obtain (i), otherwise we obtain (ii).

**Proposition 7.** Let $s = 3$ and let $t \in A$ be defined by $t(a) = d$, $t(b) = c$, $t(c) = b$, $t(d) = a$ (see Fig. 2). Then $t^2 = 1$, $A[b,c] = \langle b, c, t \rangle = \langle b, \xi \rangle \cong D_4$ and $\xi \beta \xi = \xi$.

**Proof.** $t^2 = 1$ because it fixes the vertices $a, b, c, d$. Since $t(b) = c$ we have $t\beta t^{-1} = \beta$, $(\xi \beta)^2 = \beta \xi$. Thus $\langle \xi, \beta \rangle \cong D_4$.

**Proposition 8.** Let $s = 4$ and let $t \in A$ be defined by $t(v_i) = v_{5-i}$ for $0 \leq i \leq 4$. Then $A[v_2, v_3] = \langle b, c, d, t \rangle$ and either
(i) $\xi^2 = 1$ and $A[v_2,v_3] = \langle b,\xi \rangle \cong D_8$; or 
(ii) $\xi^2 = \tilde{c}$ and $A[v_2,v_3] = \langle b,\xi \rangle$, $\xi b$ is an element of order 8 and $b(\xi b)b = (\xi b)^3 = \xi d$. Thus $A[v_2,v_3]$ is a quasidihedral group, see [7, p. 90, Satz 14.9].

In case (ii) there is no involution in $A$ interchanging $v_2$ and $v_3$.

Proof. Since $\xi(a) = e$, $\xi(b) = d$ and $\xi(c) = c$ we have $\xi a \xi^{-1} = \tilde{a}$, $\xi b \xi^{-1} = \tilde{d}$ and $\xi c \xi^{-1} = \tilde{c}$. Since $\xi^2$ fixes $v_i$ for $1 \leq i \leq 3$ we have $\xi^2 \in A(v_1,v_2,v_3) = \langle b,\tilde{c} \rangle$.

If $\xi^2 = 1$ then $\xi(v_5) = v_0$ and we have $(\xi b)^2 = \xi b \xi b = \tilde{a} \tilde{b}$, $(\xi b)^4 = (\tilde{a} \tilde{b})^2 = \tilde{c}$ and hence $\xi b$ has order 8. Hence $\langle b,\xi \rangle \cong D_8$ and this must be the whole group $A[v_2,v_3]$.

Now, assume that $\xi^2 \neq 1$. Then $\xi(v_5) = v_7$ and hence $\xi^2$ moves $v_0$, and similarly it moves $v_4$.

If $\xi^2 = \tilde{c}$ we find that $(\xi b)^2 = \xi b \xi b = \xi b \xi^{-1} \xi b = \tilde{d} \tilde{b} = bd$ and again $(\xi b)^4 = \tilde{c}$, $\xi b$ has order 8. We compute that

$$b(\xi b)b = \tilde{b} = (\xi a \xi^{-1}) \xi = \xi d,$$

$$(\xi b)^3 = \xi b (\xi b)^2 = \xi b + \tilde{b}d = \xi d.$$

Hence $b$ and $\xi b$ do not commute and the group $\langle b,\xi \rangle = \langle \tilde{b},\xi b \rangle$ has order 16. Therefore it is equal to $A[v_2,v_3]$.

The group $\langle \tilde{b},\xi \rangle$ in case (ii) has precisely 5 involutions and all of them lie in the subgroup $A(v_2,v_3) \cong D_4$. Hence every involution in $A[v_2,v_3]$ fixes $v_2$ and $v_3$. 


Proposition 9. Let $s = 5$ and define $\xi \in A$ by $\xi(a) = f$, $\xi(b) = e$, $\xi(c) = d$, $\xi(d) = c$, $\xi(e) = b$ and $\xi(f) = a$, see Fig. 4. Then $\xi^2 = 1$, $A[cd] = A(c,d) \times \langle \xi \rangle$

where $\xi$ acts on $A(c,d) = \langle b, c, d, e \rangle$ by

$$
\xi b e = e, \quad \xi c e = d, \quad \xi d e = c, \quad \xi e e = b.
$$

Proof. $\xi^2 = 1$ because $\xi^2$ fixes the vertices $a, b, c, d, e, f$. All claims are then obvious.

Let $A$ be $s$-regular ($1 \leq s \leq 5$). We say that $A$ is of type $S'$ if there exists an involution $\alpha \in A$ which flips an edge. Otherwise we must have $s = 2$ or $4$ and we say that $A$ is of type $S''$. 
5. **Conjugacy classes of shuntings**

Let $G$ and $A$ be as before, $1 \leq s \leq 5$.

**Lemma 1.** There are precisely 2 orbits for the action of $A$ on $(s+1)$-arcs. If $S_1$ and $S_2$ are two distinct $(s+1)$-arcs and $S_1(i) = S_2(i)$ for $0 \leq i \leq s$ then they are not in the same orbit.

**Proof.** Let $S_1$ and $S_2$ be as in the Lemma. If $S$ is any $(s+1)$-arc then there exists $a \in A$ such that $a(S(i)) = S_1(i) = S_2(i)$ for $0 \leq i \leq s$. Hence $a \circ S = S_1$ or $S_2$ and there are at most 2 orbits. But $S_1$ and $S_2$ must be in different orbits because $A$ is $s$-regular.

To each $(s+1)$-arc $S$ we shall associate an $A$-shunting $sh(S) = (a,\alpha)$ by taking $a = S(0)$ and defining $\alpha$ to be the unique element of $A$ such that $\alpha(S(i)) = S(i+1)$ for $0 \leq i \leq s$.

**Lemma 2.** There are precisely 2 $A$-conjugacy classes of $A$-shuntings. Two $A$-shuntings with overlap $s$ are not conjugate in $A$.

**Proof.** The map $sh$ from $(s+1)$-arcs to $A$-shuntings is bijective. It is also $A$-equivariant, i.e., we have

$$\alpha \cdot sh(S) = sh(\alpha \circ S)$$

for all $\alpha \in A$ and all $(s+1)$-arcs $S$.

Hence both assertions of Lemma 2 follow from the corresponding assertions of Lemma 1.

**Lemma 3.** If an $A$-shunting $(a,\alpha)$ is $A$-conjugate to its opposite $(a,\alpha^{-1})$ then the same is true for all $A$-shuntings.
Proof. Let us write ~ to indicate that two A-shuntings are conjugate in A.

Let \((a, \beta)\) be the unique A-shunting having overlaps with \((a, \alpha)\). By hypothesis \((a, \alpha) \sim (a, \alpha^{-1})\) and hence if \(b = a^5\) we also have \((a, \alpha) \sim (b, \alpha^{-1})\). Let \(\gamma \in A\) be such that \(\gamma \cdot (a, \alpha) = (b, \alpha^{-1})\), i.e., \(\gamma(a) = b\) and \(\gamma \alpha^{-1} = \alpha^{-1}\). Since \((a, \alpha)\) and \((a, \beta)\) have overlaps the same is true for \(\gamma \cdot (a, \alpha) = (b, \alpha^{-1})\) and \(\gamma \cdot (a, \beta)\). But \((b, \beta^{-1})\) is the unique A-shunting having overlaps with \((b, \alpha^{-1})\) and hence \(\gamma \cdot (a, \beta) = (b, \beta^{-1})\). Thus \((a, \beta) \sim (b, \beta^{-1})\).

If \((c, \sigma)\) is any A-shunting then for every \(\theta \in A\) the A-shuntings \(\theta \cdot (c, \sigma)\) and \(\theta \cdot (c, \sigma^{-1})\) are opposite. Hence if we choose \(\theta\) so that \(\theta(\sigma^i(c)) = \alpha^i(a)\) for \(0 \leq i \leq s\) then either \(\theta \cdot (c, \sigma) = (a, \alpha)\) and \(\theta \cdot (c, \sigma^{-1}) = (a, \alpha^{-1})\) or \(\theta \cdot (c, \sigma) = (a, \beta)\) and \(\theta \cdot (c, \sigma^{-1}) = (a, \beta^{-1})\).

In the first case we have
\[
(c, \sigma) \sim (a, \alpha) \sim (a, \alpha^{-1}) \sim (c, \sigma^{-1})
\]
and similarly in the second case.

We shall say that \(A\) is of the first (resp., second) kind if every A-shunting is (resp. is not) A-conjugate to its opposite.

Tutte [14] has shown that 1-regular groups are of the second kind. This is included in the following proposition.

Proposition 10. Groups of type \(2', 3', 4', 5'\) are of the first kind and groups of type \(1', 2'', 4''\) are of the second kind.

Proof. If \(A\) is of type \(1'\) let \(\{u, v\}\) be an edge of \(G\), \(G(u) = \{a, b, v\}\), and \(G(v) = \{u, x, y\}\). There is a \(\rho \in A\) such that \(\rho(v) = a,\)
\( \rho(a) = b, \rho(b) = v \) and \( a \in A \) such that \( \xi(u) = v, \xi(v) = u \). Then 
\[ \xi^2 = \rho^3 = 1 \] and \( (u, \xi \rho) \) is a shunting because \( \xi \rho(u) = \xi(u) = v \) and \( \xi \rho(v) = \xi(a) = x \) or \( y \). Its opposite shunting is \( (u, \rho^{-1} \xi) \). Since 
\[ \rho \cdot (u, \xi \rho) = (u, \rho \xi) \] and \( (u, \rho^{-1} \xi) \) have overlap 1, Lemma 2 implies that 
they are not \( A \)-conjugate and so \( A \) is of the second kind.

Now let \( A \) be of type \( 2', 3', 4', \) or \( 5' \). Let \( u \) be a vertex of \( G \) and \( \alpha \in A(u) \) an involution. Choose a vertex \( a \) so that \( \alpha(a) \neq a \)
and the distance \( d(u, a) \) is minimal. Some neighbor, say \( b \), of \( a \) is
fixed by \( \alpha \). There is an involution \( \beta \in A \) which flips the edge \( \{a, b\} \).
Then \( (b, \alpha \beta) \) is a shunting because \( \alpha \beta(a) = \alpha(b) = b \) and \( \alpha \beta(b) = \alpha(a) \neq a \).
Since \( \alpha \cdot (b, \alpha \beta) = (b, \beta \alpha) \) we see that \( (b, \alpha \beta) \) is \( A \)-conjugate to its opposite
shunting \( (b, \beta \alpha) \). Thus \( A \) is of the first kind.

Next, let \( A \) be of type \( 2" \). We shall use notation from Fig. 1
and the element \( \xi \) defined in Prop. 6, \( \xi^2 = \mathcal{B} \). Then \( (x, \xi \mathcal{A}) \) and \( (x, \xi^{-1} \mathcal{A}) \)
are two shunttings with overlap 2. Indeed, we have
\[
\xi \mathcal{A}(x) = \xi(z) = y, \quad \xi \mathcal{A}(y) = \xi(y) = z;
\]
\[
\xi^{-1} \mathcal{A}(x) = \xi^{-1}(z) = y, \quad \xi^{-1} \mathcal{A}(y) = \xi^{-1}(y) = z.
\]
Since \( \mathcal{A} \cdot (x, \xi \mathcal{A}^{-1}) = (z, \xi^{-1} \mathcal{A}) \) and \( (\mathcal{A} \xi)^2 \cdot (z, \xi^{-1} \mathcal{A}) = (x, \xi^{-1} \mathcal{A}) \), Lemma 2
implies that \( (x, \xi \mathcal{A}) \) and its opposite \( (x, \xi^{-1} \mathcal{A}) \) are not \( A \)-conjugate. Thus
\( A \) is of the second kind.

Finally, let \( A \) be of type \( 2" \). Now we shall use Fig. 3 and the
element \( \xi \) defined in Prop. 8, \( \xi^2 = \mathcal{C} \). The shunttings \( (v_0, \xi \mathcal{A} \mathcal{A}) \) and
\( (v_0, \xi^{-1} \mathcal{A} \mathcal{A}) \) have overlap 4 because
\[ \xi \overline{a} \overline{a}(v_0) = \xi \overline{a} \overline{d}(v_0) = \xi(v_4) = v_1, \]
\[ \xi \overline{a} \overline{a}(v_1) = \xi \overline{a} \overline{d}(v_1) = \xi \overline{a}(v_6) = \xi(v_3) = v_2, \]
\[ \xi \overline{a} \overline{a}(v_2) = \xi(v_2) = v_3, \]
\[ \xi \overline{a} \overline{a}(v_3) = \xi \overline{a} \overline{d}(v_6) = \xi \overline{a}(v_1) = \xi(v_1) = v_4, \]

and similarly \( \xi^{-1} \overline{a} \overline{d}(v_i) = v_{i+1} \) for \( 0 \leq i \leq 3 \). Only the step \( \overline{a} \overline{d}(v_0) = v_4 \) needs a justification. Since \( \overline{a} \overline{d}(v_1) = \overline{a}(v_6) = v_3 \) it follows that \( \overline{a} \overline{d}(v_0) \) is a neighbor of \( v_3 \). If we put \( x = \tilde{a} \overline{d}(a) \) then \( \tilde{x} = (\tilde{a} \overline{d}) \overline{a} \overline{d}(a) \), i.e., \( \tilde{x} = \tilde{d} \). Since \( v_3 \) is a vertex of \( x \) this forces \( x = d \) and hence \( \overline{a} \overline{d}(v_0) = v_4 \).

Hence \( \overline{a} \overline{a} \cdot (v_0, \overline{a} \overline{a} \xi) = (v_4, \xi \overline{a} \overline{a}) \) and \( (\xi \overline{a} \overline{a})^4 \cdot (v_0, \xi \overline{a} \overline{a}) = (v_4, \xi \overline{a} \overline{a}) \). Now the argument is the same as for the type 2".

The proof is completed.
6. Shuntings as generators

Here, \( G = (V,E) \) is a connected cubic graph.

**Proposition 11.** Let \((a,\alpha)\) and \((a,\beta)\) be two shuntings with overlap \( s \geq 1 \). Then the group \( A = \langle \alpha \beta \rangle \) is \( s \)-transitive.

For the proof see [1, Theorem 17.5, p. 115].

**Proposition 12.** Let \( A \) be an \( s \)-regular subgroup of \( \text{Aut}(G) \), \( 1 \leq s \leq 5 \). Then \( A \) is generated by any two \( A \)-shuntings with overlap \( s \).

This is an immediate consequence of Proposition 11.

**Proposition 13.** Let \((a,\alpha)\) be a shunting in \( \Gamma_3 \). Then

(i) there exists an involution \( \beta \in \text{Aut}(\Gamma_3) \) such that \( A = \langle \alpha, \beta \rangle \) is \( \omega \)-transitive;

(ii) there exists a shunting \((a,\gamma)\) having overlap 1 with \((a,\alpha)\) and such that \( B = \langle \alpha, \gamma \rangle \) is \( \omega \)-transitive.

**Proof.** (i) Let \( a_i = a^i(a) \) for \( i \in \mathbb{Z} \). Choose an involution \( \beta \) such that \( \beta(a_i) = a_i \) for \( 0 \leq i \leq 6 \), \( \beta(a_{-1}) \neq a_{-1} \), and \( \beta(a_7) \neq a_7 \). Then the shuntings \((a,\alpha)\) and \( \beta \cdot (a,\alpha) = (a,\beta\alpha\beta) \) have overlap 6. By Proposition 11 the group \( \langle \alpha, \beta\alpha\beta \rangle \) is 6-transitive. Since it cannot be 6-regular it follows [12, p. 63] that it is 7-transitive. Continuing in this way we obtain that \( \langle \alpha, \beta\alpha\beta \rangle \) (and also \( A \)) is \( \omega \)-transitive.

(ii) One can choose a shunting \((a,\gamma)\) so that it has overlap 1 with \((a,\alpha)\) and, moreover, that \((\alpha(a),\alpha)\) and \( \gamma \cdot (a,\alpha) = (\alpha(a),\gamma\alpha\gamma^{-1}) \) have overlap 6. Then \( \langle \alpha, \gamma\alpha\gamma^{-1} \rangle \) (and also \( B \)) is \( \omega \)-transitive.
Proposition 14. Let $A$ be an $s$-regular subgroup of $\text{Aut}(G)$, $2 \leq s \leq 5$. Assume that $B$ is a proper normal subgroup of $A$ and that there is a shunting $(a, \beta)$ with $\beta \in B$. Then $(A:B) = 2$ and $B$ is $(s-1)$-regular.

Proof. Since $A$ is $s$-regular there exists a unique element $\alpha \in A$ which fixes $\beta^i(a)$ for $0 \leq i \leq s-1$ and $\alpha \neq 1$. Since $\alpha^2$ fixes also $\beta^s(a)$ we have $\alpha^2 = 1$. Since $\alpha \neq 1$ the $s$-regularity of $A$ implies that $\alpha$ moves $\beta^{-1}(a)$ and $\beta^s(a)$. Therefore the shunting $\alpha \cdot (a, \beta) = (a, \alpha \beta a)$ has overlap $s-1$ with $(a, \beta)$. Since $\beta \in B$, $\alpha \in A$ and $B \triangleleft A$ it follows that $(a, \alpha \beta a)$ is also a $B$-shunting. Now, Prop. 11 implies that $B$ is $(s-1)$-transitive. Since $B \neq A$, $B$ is $(s-1)$-regular and so $(A:B) = 2$. 
7. Some Important Amalgams

An amalgam (more precisely, group amalgam) is an ordered pair \((X,Y)\) consisting of two groups \(X\) and \(Y\) such that

(i) \(X \cap Y\) is a subgroup in both \(X\) and \(Y\),

(ii) \(X\) and \(Y\) induce on \(X \cap Y\) the same group structure.

Let \((X,Y)\) and \((X',Y')\) be two amalgams. A homomorphism \(f: (X,Y) \to (X',Y')\) is a map \(f: X \cup Y \to X' \cup Y'\) such that \(f(X) \subseteq X'\), \(f(Y) \subseteq Y'\) and the restrictions \(f_X: X \to X'\) and \(f_Y: Y \to Y'\) of \(f\) are group homomorphisms. A bijective homomorphism \(f: (X,Y) \to (X',Y')\) is called an isomorphism.

Now we describe several important amalgams.

Amalgam 1': \((X_1,Y_1)\) where \(X_1 = \langle a; a^3 = 1 \rangle\), \(Y_1 = \langle y; y^2 = 1 \rangle\), \(X_1 \cap Y_1 = \{1\}\).

Amalgam 2': \((X_2,Y_2)\) where \(X_2 = \langle a,b; a^2 = b^2 = (ab)^3 = 1 \rangle \cong D_3\), \(Y_2 = \langle b,y; b^2 = y^2 = (yb)^2 = 1 \rangle \cong C_2 \times C_2\) and \(X_2 \cap Y_2 = \langle b \rangle \cong C_2\).

Amalgam 2": \((X_2',Y_2')\) where \(X = \langle a,b; a^2 = b^2 = (ab)^3 = 1 \rangle \cong D_3\), \(Y'' = \langle y; y^4 = 1 \rangle\), \(y^2 = b\), \(X_2 \cap Y_2'' = \langle b \rangle \cong C_2\).

Amalgam 3': \((X_3,Y_3)\) where \(X_3 = \langle a,b,c; a^2 = b^2 = c^2 = (ab)^2 = (bc)^2 = (ac)^3 = 1 \rangle \cong D_6\), \(Y_3 = \langle b,c,y; b^2 = c^2 = y^2 = (bc)^2 = 1, yby = c \rangle \cong D_4\), and \(X_3 \cap Y_3 = \langle b,c \rangle \cong C_2 \times C_2\).

Amalgam 4': \((X_4,Y_4')\) where \(X_4\) is generated by the elements \(a,b,c,d\) with defining relations

\[
\begin{align*}
a^2 &= b^2 = c^2 = d^2 = (ab)^2 = (bc)^2 = (cd)^2 = 1, \\
(ac)^2 &= b, \quad (bd)^2 = c, \quad (ad)^3 = 1,
\end{align*}
\]

and \(Y_4\) is generated by the elements \(b,c,d,y\) with defining relations...
\[ b^2 = c^2 = d^2 = y^2 = (bc)^2 = (cd)^2 = (yc)^2 = 1, \ yby = d. \]

Then \( X_4 \cong S_4, \ Y_4 \cong D_8 \) and \( X_4 \cap Y_4 = \langle b, c, d \rangle \cong D_4 \), i.e., \( X_4 \cap Y_4 \) is a Sylow 2-subgroup of \( X_4 \).

**Amalgam \( 4'' \):** \((X_4, Y_4)\) where \( X_4 \) is the same as in the previous case and \( Y_4 \) is generated by \( b, c, d, y \) with defining relations

\[
\begin{align*}
\text{Amalgam } 4'' & : (X_4, Y_4) \\
\quad & \text{where } X_4 \text{ is the same as in the previous case and } Y_4 \text{ is generated by } b, c, d, y \text{ with defining relations} \\
\quad & b^2 = c^2 = d^2 = (bc)^2 = (cd)^2 = 1, \quad (bd)^2 = c, \\
\quad & y^2 = c, \quad yby^{-1} = d.
\end{align*}
\]

Then \( Y_4 \cong D_8 \) (quasidihedral group of order 16) and \( X_4 \cap Y_4 = \langle b, c, d \rangle \cong D_4 \).

**Amalgam \( 5''' \):** \((X_5, Y_5)\) where \( X_5 \) is generated by the elements \( a, b, c, d, e \) with defining relations

\[
\begin{align*}
\text{Amalgam } 5''' & : (X_5, Y_5) \\
\quad & \text{where } X_5 \text{ is generated by the elements } a, b, c, d, e \text{ with defining relations} \\
\quad & a^2 = b^2 = c^2 = d^2 = e^2 = (ab)^2 = (bc)^2 = (cd)^2 = (de)^2 = (ac)^2 = (bd)^2 = (ce)^2 = 1, \\
\quad & (ad)^2 = bc, \quad (be)^2 = cd, \quad (ae)^3 = 1,
\end{align*}
\]

and \( Y_5 \) is generated by the elements \( b, c, d, e, y \) with defining relations

\[
\begin{align*}
\text{Amalgam } 5''' & : (X_5, Y_5) \\
\quad & \text{where } X_5 \text{ is generated by the elements } b, c, d, e, y \text{ with defining relations} \\
\quad & b^2 = c^2 = d^2 = e^2 = y^2 = (bc)^2 = (cd)^2 = (de)^2 = (bd)^2 = (ce)^2 = 1, \\
\quad & (be)^2 = cd, \quad yby = e, \quad ycy = d.
\end{align*}
\]

In this case

\[
\begin{align*}
X_5 & = \langle a, bc, cd, e \rangle \times \langle c \rangle, \\
\langle a, bc, cd, e \rangle & \cong S_4, \quad \langle c \rangle \cong C_2,
\end{align*}
\]

and

\[
\begin{align*}
Y_5 & = \langle b, c, d, e \rangle \rtimes \langle y \rangle.
\end{align*}
\]

We have \( X_5 \cap Y_5 = \langle b, c, d, e \rangle = \langle b, e \rangle \times \langle c \rangle = \langle b, e \rangle \times \langle d \rangle \cong D_4 \times C_2. \)
We shall only prove the above claim for $X_5$. The subgroup $X' = \langle b, c, d \rangle$ is normal in $X_5$ because the defining relations for $X_5$ give $aba = b$, $aca = c$, $ada = bcd$, $ebe = bcd$, $ece = c$, $ede = d$ and hence $a$ and $e$ normalize $X'$. Since $b, c, d$ commute and have at most order 2 each, the group $X'$ is elementary abelian of order 1, 2, 4 or 8. Since $a^2 = e^2 = 1$ and $(ae)^3 = 1$ it follows that the group $X'' = \langle a, e \rangle$ is a homomorphic image of $D_3$. Since $X_5 = X'X''$ we have $|X_5| \leq 48$. In order to prove that $X_5 \cong S_4 \times C_2$ it suffices to construct a surjective homomorphism $X_5 \to S_4 \times C_2$. There is such a homomorphism which sends

$$a \to (12), \quad bc \to (12)(34), \quad cd \to (13)(24), \quad e \to (13)$$

and $c$ to the generator of $C_2$. In order to verify this claim one has just to check that the images of the generators $a, b, c, d, e$ satisfy the defining relations for $X_5$ which is routine. Since this must be an isomorphism, $X'$ is elementary abelian of order 8, and $X'' \cong D_3$.

Since $e$ normalizes $\langle b, c, d \rangle$ the group $X''' = \langle b, c, d, e \rangle$ has order 16. The element $be$ has order 4 because $(be)^2 = cd$. Thus $\langle b, e \rangle \cong D_4$ and since neither $c$ nor $d$ is in $b, e$ we have $X''' = \langle b, e \rangle \times \langle c \rangle = \langle b, e \rangle \times \langle d \rangle \cong D_4 \times C_2$.

Let $A$ be an $s$-regular subgroup of $\text{Aut}(G)$ where $G$ is a connected cubic graph ($1 \leq s \leq 5$). To each $1$-arc $S$ of $G$, say $S(0) = u$, $S(1) = v$ we associate an amalgam $(A(u), A[u, v])$. 
Proposition 15. The amalgam \((A(u), A[u,v])\) is isomorphic to \((X_s, Y_s)\) where
if \(s = 2\) or \(4\) then \(Y_s = Y'_s\) if \(A\) is of the first kind and \(Y_s = Y''_s\)
if \(A\) is of the second kind.

Proof. This is immediate from the definition of the amalgams \((X_s, Y_s)\)
and from Propositions 6 - 9 and their proofs.
8. Regular Actions on \( \Gamma_3 \)

Let \((X_s, Y_s)\) be the amalgams constructed in the previous section where \(Y_s = Y'_s\) or \(Y'_s\) if \(s = 2\) or \(4\). Put \(H_s = X_s \cap Y_s\) and

\[
\begin{align*}
A'\_s &= X_s * H_s * Y_s & (s=1,3,5), \\
A''\_s &= X_s * Y''_s & (s=2,4), \\
A'\_s &= X_s * Y'_s & (s=2,4),
\end{align*}
\]

where * denotes the free product with amalgamation.

We shall write \(A_s\) for \(A'_s\) or \(A''_s\).

Let \(G_s\) be the graph whose vertex set is the disjoint union of \(A_s/X_s\) and \(A_s/Y_s\) and whose edge set consists of all \(\{uX_s, uY_s\}\) for \(u \in A_s\). Vertices of type \(uX_s\) have valence 3 and those of type \(uY_s\) have valence 2. It is well-known that \(G_s\) is a tree, see [10, Theorem 7, p. 1-50 and 51]. The group \(A_s\) acts on \(G_s\) by left multiplication. The fixer in \(A_s\) of the vertex \(X_s\) of \(G_s\) is the subgroup \(X_s\) of \(A_s\); similarly the fixer in \(A_s\) of the vertex \(Y_s\) is the subgroup \(Y_s\). The action of \(A_s\) on \(G_s\) is faithful because \(H_s\) is the fixer of the edge \(<X_s,Y_s>\) and the only normal subgroup of \(A_s\) which is contained in \(H_s\) is the trivial subgroup. Note that \(A_s\) is transitive on vertices of \(G_s\) of valence 3 and also on vertices of valence 2.

Let us define a new graph whose vertex set is \(A_s/X_s\) and in which two vertices are adjacent if and only if they are neighbors in \(G_s\) of one vertex of valence 2. This new graph is \(\Gamma_3\), i.e., a regular 3-valent tree. The automorphisms of \(G_s\) induce automorphisms of \(\Gamma_3\) and this establishes...
an isomorphism between $\text{Aut}(G_s)$ and $\text{Aut}(\Gamma_3)$. Hence, $A_s$ acts on $\Gamma_3$ and the action is faithful. The fixer in $A_s$ of the vertex $X_s$ is again the subgroup $X_s$ and the stabilizer of the edge $\{X_s, yX_s\}$ is the subgroup $H_s$.

We have the following

**Proposition 16.** The graph whose vertex set is $A_s/X_s$ and in which two vertices $uX_s$ and $vX_s$ are adjacent if and only if $u^{-1}v \in X_s yX_s$, is an infinite cubic tree $\Gamma_3$. $A_s$ acts on $\Gamma_3$ by left multiplication. This action is faithful and $A_s$ is an s-regular subgroup of $\text{Aut}(\Gamma_3)$.

$A'_s$ is of type $s'$ and $A''_s$ is of type $s''$.

**Proof.** For the first assertion we need to verify that $u^{-1}v \in X_s yX_s$, if and only if $uX_s$ and $vX_s$ are distinct and have a common neighbour in $G_s$. If $u^{-1}v = x_1 y x_2$ where $x_1, x_2 \in X_s$ and $y_1, y_2 \in Y_s$, then $uX$ and $vX$ are both neighbours of $ux_s y_s$ in $G_s$.

Conversely, let $uX_s$ and $vX_s$ have a common neighbour $wX_s$ in $G_s$. Then $wy_1 = ux_1$ and $wy_2 = vx_2$ for suitable $x_1, x_2 \in X_s$ and $y_1, y_2 \in Y_s$. Hence $u^{-1}v = x_1 y_1^{-1} y_2 x_2^{-1}$. If $uX_s \neq vX_s$ then $y_1^{-1} y_2 \notin X_s$. But $y_1^{-1} y_2 \in Y_s$ and so $y_1^{-1} y_2 \in Y_s$. Therefore $u^{-1}v \in X_s yX_s$.

All the other assertions are obvious from the remarks made above.

A subgroup $K$ of $A_s$ is said to be small (in $A_s$) if $K \lhd A_s$, $K \cap X_s = K \cap Y_s = 1$, and $(A_s : KX_s) > 2$. In that case $Y_s \not\subset KX_s$ and in fact $(A_s : KX_s) \geq 4$.

**Theorem 1.** Let $G$ be a connected cubic graph and $A$ an s-regular subgroup of $\text{Aut}(G)$, $1 \leq s \leq 5$. Let $A_s = A'_s$ or $A''_s$ be of the same type as $A$. Then there exists a covering morphism $(f, g) : (\Gamma_3, A_s) \to (G, A)$ and $\ker g$ is small in $A_s$. 
Conversely, let $K$ be a small subgroup of $A_s$ and put $A_K = A_s/K$.
Let $G_K$ be the graph whose vertex-set is $A_s/KX_s$ and in which two
vertices $uKX_s$ and $vKX_s$ are adjacent iff $u^{-1}v \in KX_sYX_s$. Then $G_K$ is
a connected cubic graph, $A_K$ is an $s$-regular subgroup of $\text{Aut}(G_K)$ and $A_K$
is of the same type as $A_s$. There is a covering morphism $(f,g): (\Gamma_3,A_s) \to (G_K,A_K)$ where $g: A_s \to A_K$ is the canonical map and $f: \Gamma_3 \to G_K$ is defined
by $f(uX) = uKX$.

Proof. Let $\{u,v\}$ be an edge of $G$. The amalgam $(A(u),A[u,v])$ is
isomorphic to $(X_s,Y_s)$ by Prop. 15. An isomorphism $g: (X_s,Y_s) \to (A(u),A[u,v])$
can be extended to a unique homomorphism $g: A_s \to A$, and $g$ is surjective
by Prop. 1 (ii).

If $z \in xX_s \ (x \in A_s)$ then $z = xx_1, x_1 \in X_s$ and $g(z)(u) =
g(x)g(x_1)(u) = g(x)(u)$ because $g(x_1) \in g(X_s) = A(u)$. Therefore there
exists a map $f: A_s/X_s \to G$ such that $f(xX_s) = g(x)(u)$ for all $x \in A_s$.
Let $z_1X_s$ and $z_2X_s$ be adjacent in $\Gamma_3$. Then $z_1^{-1}z_2 \in X_sYX_s$, i.e.,
$z_1^{-1}z_2 = x_1yx_2$ where $x_1,x_2 \in X_s$. We claim that $a = f(z_1X_s) = g(z_1)(u)$
and $b = f(z_2X_s) = g(z_2)(u)$ are adjacent in $G$.

For this it suffices to show that the vertices $g(z_1^{-1}z_2)(u)$ and
$g(z_1^{-1})(a) = u$ are adjacent. This follows from
$$g(z_1^{-1}z_2)(u) = g(x_1yx_2)(u) = g(x_1)g(y)(u) = g(x_1)(v),$$
where we used that $g(x_1) \in A(u)$ and that $g(y)$ flips the edge $\{u,v\}$. It
follows that $g$ is a graph homomorphism and it is now easy to check that
$(f,g)$ is a covering morphism.

Now we pass to the converse. Since $A_s \neq KX_s$ and $A_s = \langle X_s,y \rangle$
we have $y \notin KX_s$. The graph $G_K$ is connected because $KX_s$ and $y$
generate $A_s$. Since $A_s$ acts transitively on $G_K$, the graph $G_K$ is regular. The valency of $G_K$ is equal to the number of left cosets of $KX_s$ in $KX_syKX_s = KX_sX_s = X_sX_sK$. But this last number is equal to the index of $yX_syK \cap X_sK$ in $X_sK$. We claim that $yX_syK \cap X_sK = (X_syX_s)K$ and consequently $G_K$ is a cubic graph. Otherwise we would have $yX_syK \cap X_sK \neq (X_s\cap Y_s)K$ and since $(X_s\cap Y_s)K$ is a maximal subgroup of both $X_sK$ and $yX_syK$, we obtain that $yX_syK = X_sK$. This means that $y$ normalizes $KX_s$ and since $y^2 \in X_s$ we obtain $(A_s:KX_s) = 2$ which is again a contradiction.

Thus $G_K$ is a connected cubic graph and $A_s$ acts on it. Let $N$ be the kernel of the corresponding homomorphism $A_s \to \text{Aut}(G_K)$. We have $K \leq N$ and since elements of $N$ fix the vertices $KX_s$ and $yKX_s$ of $G_K$ we also have

$$N = K \cdot (N \cap X_s \cap Y_s).$$

Therefore $N = K \cdot (N \cap X_s \cap Y_s)$. The group $N \cap X_s \cap Y_s$ is normal in both $X_s$ and $Y_s$ and consequently $N \cap X_s \cap Y_s = \{1\}$, and $N = K$. Hence the group $A_K = A_s/K$ acts on $G_K$ faithfully and we can consider it as a subgroup of $\text{Aut}(G_K)$. Since the amalgam $(KX_s/K,KY_s/K)$ is isomorphic to the amalgam $(X_s,Y_s)$ we infer that $A_K$ is of the same type as $A_s$.

Now it is easy to verify that $(f,g)$ is a covering morphism.

Theorem 2. Let $A_s = A'_s$ (1 ≤ s ≤ 5) resp. $A_s = A''_s$ (s = 2 or 4) and let $A$ be any $s$-regular subgroup of $\text{Aut}(\Gamma_3)$ of the same type as $A_s$. Then $A_s$ and $A$ are conjugate in $\text{Aut}(\Gamma_3)$.

Proof. By Theorem 1 there is a covering morphism $(f,g): (\Gamma_3,A_s) \to (\Gamma_3,A)$. For $x \in A_s$ let $\phi_x: A_s/X_s \to A_s/X_s$ be the left multiplication by $x$. Then we have $f \circ \phi_x = g(x) \circ f$ for all $x \in A_s$. Since $f$ is necessarily
an automorphism of $\Gamma_3$, we have $g(x) = f \circ \phi_x \circ f^{-1} = fxf^{-1}$. Hence $g: A_5 \to A$ is an isomorphism obtained by restricting the inner automorphism of $\text{Aut}(\Gamma_3)$ induced by $f$.

**Proposition 17.** Let $K$ be a small subgroup of $A_5 = A_5'$ or $A_5''$. Then $G_K$ is bipartite iff $K \subseteq A_5^+$. 

**Proof.** Since $(A_K)^+ = A_5^+ K/K$, the claim follows from Proposition 1 (iv).
9. Regular Subgroups of Regular Groups

It is easy to check that the following are embeddings among the amalgams defined in section 7:

\[ f_{12} : (X_1, Y_1) + (X_2, Y_2); \]
\[ f_{12}(a) = ab, \ f_{12}(y) = y; \]

\[ f'_{23} : (X_2, Y_2') + (X_3, Y_3), \]
\[ f'_{23}(a) = ab, \ f'_{23}(b) = bc, \ f'_{23}(y) = y; \]

\[ f''_{23} : (X_2, Y_2'') + (X_3, Y_3), \]
\[ f''_{23}(a) = ab, \ f''_{23}(b) = bc, \ f''_{23}(y) = zb; \]

\[ f_{14} : (X_1, Y_1') + (X_4, Y_4'), \]
\[ f_{14}(a) = ad, \ f_{14}(y) = y; \]

\[ f'_4 : (X_4, Y_4') + (X_5, Y_5), \]
\[ f'_4(a) = ab, \ f'_4(b) = bc, \ f'_4(c) = cd, \ f'_4(d) = de, \]
\[ f'_4(y) = y; \]

\[ f''_4 : (X_4, Y_4'') + (X_5, X_5), \]
\[ f''_4(a) = ab, \ f''_4(b) = bc, \ f''_4(c) = cd, \ f''_4(d) = de, \]
\[ f''_4(y) = yc. \]

These embeddings extend respectively to the embeddings \( A_1' \rightarrow A_2', \ A_2' \rightarrow A_3', \ A_2'' \rightarrow A_3', \ A_1' \rightarrow A_4', \ A_4' \rightarrow A_5' \) and \( A_4'' \rightarrow A_5'. \)
Let $G$ be a connected cubic graph and $A$ an $s$-regular subgroup of $\text{Aut}(G)$, $1 \leq s \leq 5$. Assume that $A$ has a $t$-regular subgroup.

**Theorem 3.** Let $G, A, B$ be as above with $1 \leq t < s \leq 5$. The possible types for $A$ and $B$ are indicated by "Yes" in the table below.

<table>
<thead>
<tr>
<th>$B$</th>
<th>$2'$</th>
<th>$2''$</th>
<th>$3'$</th>
<th>$4'$</th>
<th>$4''$</th>
<th>$5'$</th>
</tr>
</thead>
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<tr>
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<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>$2'$</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>$2''$</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>$3'$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td></td>
<td>No</td>
</tr>
<tr>
<td>$4'$</td>
<td>Yes</td>
<td></td>
<td></td>
<td></td>
<td>Yes</td>
<td></td>
</tr>
<tr>
<td>$4''$</td>
<td>Yes</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Proof.** Since the groups $A_s'$ and $A_s''$ are $s$-regular on $\Gamma_3$, the examples for all "Yes" entries are provided by $G = \Gamma_3$.

Let $S$ be a 3-arc in $G$ and put $S(0) = x$, $S(1) = y$, $S(2) = z$, $S(3) = u$. Let $A_0(y,z)$ be the subgroup of $A(y,z)$ consisting of those $a \in A(y,z)$ which satisfy $a(x) \neq x \Rightarrow a(u) \neq u$. If $s = 4$ then using the notation of Proposition 4 and taking $v_1 = y$, $v_2 = z$ we obtain $A_0(y,z) = \langle ac \rangle$. If $s = 5$ then using the notation of Proposition 5 and taking $y = b$, $z = c$ we obtain $A_0(y,z) = \langle \alpha d, \delta \rangle$. In both cases we see that if $a \in A_0(y,z)$ and $a(x) \neq x$ then $a$ has order 4.
If $t = 2$ then $B_0(y,z) = \langle \alpha \rangle$ where $\alpha$ has order 2 and $\alpha(x) \neq x$. If $t = 3$ then there exists an $\alpha \in B_0(y,z)$ such that $\alpha(x) \neq x$. Since also $\alpha(u) \neq u$, it follows from 3-regularity of $B$ that $\alpha^2 = 1$. Hence, in both cases there exists an $\alpha \in B_0(y,z)$ such that $\alpha(x) \neq x$ and $\alpha$ has order 2.

Since $B_0(y,z)$ is a subgroup of $A_0(y,z)$ it follows that $(s,t) \neq (5,2), (5,3), (4,2)$ or $(4,3)$.

If $A$ is of type 2" or 4" then $B$ cannot be of type 1'; this is immediate from the definition of these types.

We have taken care of all "No" entries in our table.

**Corollary.** Let $G$ be a connected cubic graph and assume that $\text{Aut}(G)$ has two subgroups $A$ and $B$ such that $A$ is $s$-regular, $s = 2$ or 3, and $B$ is $t$-regular, $t = 4$ or 5.

Then $G = \Gamma_3$.

**Proof.** Since $\text{Aut}(G)$ is $t$-transitive but not $t$-regular it follows that it is $(t+1)$-transitive. This argument can be repeated ad infinitum and hence $\text{Aut}(G)$ is $\omega$-transitive. Therefore $G$ cannot contain any circuits, i.e., $G$ is a tree.

**Theorem 4.** Let $A$ be an $s$-regular subgroup of $\text{Aut}(\Gamma_3)$. Then we have:

(i) If $A$ is of type 2', 3' or 5' then it has precisely two $(s-1)$-regular subgroups, say, $B'$ and $B"$. If $s = 3$ or 5 then $B'$, say, is of the first kind and $B"$ of the second kind. If $(a, \alpha)$ is a shunting with $\alpha \in A$ then either $\alpha \in B'$ and $\alpha \notin B"$ or $\alpha \in B"$ and $\alpha \notin B'$. Two $A$-shunttings $(a, \alpha)$ and $(b, \beta)$ are $A$-conjugate
of and only if $\alpha$ and $\beta$ are both in $B'$ or both in $B''$.

(ii) If $A$ is of type $4'$ then it has precisely 16 1-regular subgroups which split into two conjugacy classes, each of size 8. Let $B'$ resp. $B''$ be the set-theoretic union of the 1-regular subgroups in the first resp. second conjugacy class. If $(a,\alpha)$ is a shunting with $a \in A\setminus A_2$ then either $a \in B'$ and $a \notin B''$ or $a \in B''$ and $a \notin B'$. Two $A$-shuntings $(a,\alpha)$ and $(b,\beta)$ are $A$-conjugate if and only if $\alpha$ and $\beta$ belong both to $B'$ or both to $B''$.

Proof. (i) We saw in the previous section that $A_5$ contains $A_4$ and $A_4''$ as subgroups, $A_3$ contains $A_2'$ and $A_2''$ as subgroups, and $A_2'$ contains $A_1$ as a subgroup. In fact we have two subgroups of $A_2'$ isomorphic to $A_1'$ namely $<ab,y>$ and $<ab,yb>$. We denote for $s = 2,3,5$ these two subgroups by $B'$ and $B''$.

Since $B'$ is generated by two shuntings it follows that there exists a shunting $(a,\alpha)$ with $\alpha \in B' \setminus B''$. Similarly, there exists shunting $(b,\beta)$ with $\beta \in B'' \setminus B'$. Since $(A:B') = (A:B'') = 2$ it follows that $(a,\alpha)$ and $(b,\beta)$ are not $A$-conjugate. Lemma 1 then implies that if $(c,\gamma)$ is an $A$-shunting then either $\gamma \in B' \setminus B''$ or $\gamma \in B'' \setminus B'$. It follows that a $B'$-shunting and a $B''$-shunting cannot be $A$-conjugate. Consequently, any two $B'$-shuntings are conjugate in $A$ and similarly any two $B''$-shuntings.

Now, let $B$ be any $(s-1)$-regular subgroup of $A$. If $(c,\gamma)$ is a $B$-shunting then $(c,\gamma)$ is either $A$-conjugate to $(a,\alpha)$ or to $(b,\beta)$. In the first case we obtain $B' \subseteq B$ and in the second case $B'' \subseteq B$. Thus $B = B'$ or $B''$. 
(ii) We know that \( A \) has a 1-regular subgroup, say \( B'_1 \). By Proposition 14, \( B'_1 \) is not normal in \( A \). By Theorem 3, \( B'_1 \) is a maximal subgroup of \( A \) and hence it is its own normalizer in \( A \). Therefore \( B'_1 \) has 8 conjugates in \( A \) which we denote by \( B'_{1i} \), \( 1 \leq i \leq 8 \). By Theorem 3 there is a 5-regular subgroup \( H' \) of \( \text{Aut}(\Gamma_3) \) containing \( A \). Again \( B'_1 \) is its own normalizer in \( H \) since \( H \) contains no 2-regular subgroups. Hence \( B'_{1i} \) has 16 conjugates in \( H \) and all these conjugates lie in \( A \). These conjugates include \( B'_{1i} \), \( 1 \leq i \leq 8 \), and we denote the remaining 8 by \( B''_{1i} \), \( 1 \leq i \leq 8 \).

Now, let \( B \) be any 1-regular subgroup of \( A \). Let us use the notation from Fig. 3. Since \( B(v_2) \cong C_3 \) and \( A(v_2) \cong S_4 \) it follows that there are at most four choices for \( B(v_2) \). There is an element \( \alpha \in B \) of order 2 such that \( \alpha(v_2) = v_3 \) and \( \alpha(v_3) = v_2 \). Hence \( \alpha \in A[v_2, v_3] \) and using Proposition 8 (i) we find that \( \alpha \) must be one of the following \( \xi, \xi \xi \delta, \xi \xi \xi \delta \) or \( \xi \delta \xi \delta \delta \delta \). Since \( B = \langle B(v_2), \alpha \rangle \) by Proposition 1 (ii), we see that there are at most 16 \( 1 \)-regular subgroups in \( A \). Hence there are precisely 16 such subgroups.

Let \( (a, a) \) be a shunting with \( a \in B'_1 \). Let \( (a, \beta) \) be a shunting with \( \beta \in A \) having overlap 4 with \( (a, a) \). Since \( (a, \beta) \) is \( H \)-conjugate to \( (a, a) \) by (i), we see that every \( A \)-shunting \( (c, \gamma) \) is such that \( \gamma \) belongs to a \( 1 \)-regular subgroup of \( A \). All the remaining assertions will follow when we show that any two \( B'_1 \)-shunttings are \( A \)-conjugate. Let \( (a, \alpha) \) and \( (a, \beta) \) be two shunttings of overlap 1 in, say, \( B'_1 \). By Proposition 10, \( B'_1 \) is of the second kind. Thus, \( (a, \alpha) \) is
B₁'-conjugate to \((a, \beta^{-1})\). Again by Proposition 10, \(A_4\) is of the first kind and, therefore, \((a, \beta^{-1})\) is \(A\)-conjugate to \((a, \beta)\). So \((a, a)\) is in fact \(A\)-conjugate to \((a, \beta)\).
10. Centralizers and Normalizers

**Theorem 5.** Let $A$ be an $s$-regular subgroup of $\text{Aut}(\Gamma_3)$, $1 \leq s \leq 5$. Then the centralizer of $A$ in $\text{Aut}(\Gamma_3)$ is trivial.

**Proof.** Let $\alpha$ be a nonidentity element of $\text{Aut}(\Gamma_3)$, and $x$ a vertex of $\Gamma_3$ not fixed by $\alpha$, say $\alpha(x) = y$. Since $A$ is arc transitive there exists a rotation about $x$ in $A$, say, $p$. Now, $p$ only fixes $x$ since the graph is $\Gamma_3$. Since $\alpha p \alpha^{-1}(y) = y$, $\alpha p \alpha^{-1} \neq p$. Thus $\alpha$ is not in the centralizer of $A$.

**Theorem 6.** Let $A$ be an $s$-regular subgroup of $\text{Aut}(\Gamma_3)$, $1 \leq s \leq 5$. We have

(i) If $s = 3$ or $5$ then $A$ is its own normalizer in $\text{Aut}(\Gamma_3)$;
(ii) If \( s = 1, 2 \) or 4 then the normalizer of \( A \) in \( \text{Aut}(\Gamma_3) \) is the unique \((s+1)\) - regular subgroup of \( \text{Aut}(\Gamma_3) \) containing \( A \).

**Proof.** (i) Let \( B' \) resp., \( B'' \) be the \((s-1)\) - regular subgroups of \( A \) of the first resp., second kind (Theorem 4). Let \((a, \alpha)\) and \((a, \beta)\) be two \( A \)-shuntings with overlap \( s \) and so \( A = \langle \alpha, \beta \rangle \). Let \( S \) be the \( s \)-arc \( S(i) = \alpha^i(a) = \beta^i(a), \ \) for \( 0 \leq i \leq s \). Let \( \gamma \in \text{Aut}(\Gamma_3) \) normalize \( A \). We can choose \( \delta \in A \) such that \( \delta \gamma \delta^{-1} = S \). Hence if \( \sigma = \delta \circ \gamma \) then either \( \sigma \alpha^{-1} = \alpha \) and \( \sigma \beta^{-1} = \beta \) or \( \sigma \alpha^{-1} = \beta \) and \( \sigma \beta^{-1} = \alpha \). The latter is impossible because \( \sigma \beta^{-1} \) is of the first kind and so \( \sigma \beta^{-1} = B' \). Hence the first alternative holds and consequently \( \sigma \) centralizes \( A \). By Theorem 5 we have \( \sigma = 1 \), i.e., \( \gamma = \delta^{-1} \in A \).

(ii) We know that there exists an \((s+1)\)-regular subgroup \( H \) of \( \text{Aut}(\Gamma_3) \) containing \( A \). Let \((a, \alpha), (a, \beta), S, \gamma, \delta \) and \( \sigma \) be as in (i). If \( \sigma \alpha^{-1} = \beta \) and \( \sigma \beta^{-1} = \alpha \) we choose \( \varepsilon \in H \) such that \( \varepsilon \alpha \varepsilon^{-1} = \beta \) and \( \varepsilon \beta \varepsilon^{-1} = \alpha \). Then \( \varepsilon \sigma \) commutes with \( \alpha \) and \( \beta \) and we can finish the proof in the same way as in (i).

**Proposition 18.** Let \( B \) be a 1-regular subgroup of \( \text{Aut}(\Gamma_3) \). Then there are precisely two 4-regular subgroups (both of the first kind) containing \( B \).

**Proof.** By Theorems 2 and 4 we know that there exists a subgroup \( A \) of \( \text{Aut}(\Gamma_3) \) of type \( 4' \) containing \( B \). Let \( H \) be the normalizer of \( B \) in \( \text{Aut}(\Gamma_3) \). Then \( H \) is 2-regular and \( H \cap A = B \). Hence if \( \alpha \in H \setminus B \) then \( \alpha A \alpha^{-1} \neq A \). We claim that \( A \) and \( \alpha A \alpha^{-1} \) are the only 4-regular subgroups containing \( B \).

Let \( K \) be such a subgroup. By Theorem 3, \( K \) is of the first kind. By Theorem 2 there is a \( \beta \in \text{Aut}(\Gamma_3) \) such that \( \beta K \beta^{-1} = A \). Let
L be the normalizer of A in Aut(\(\Gamma_3\)). Then L is 5-regular and by Theorem 4 (ii) there is a \(\gamma \in L\) such that \(\gamma \beta \beta^{-1} \gamma^{-1} = B\). Thus \(\delta = \gamma \beta \in H\) and \(K = \beta^{-1}A \beta = \delta^{-1} \gamma A \gamma^{-1} \delta = \delta^{-1}A \delta\), and so \(K = A\) or \(\alpha A \alpha^{-1}\).

On Fig. 5 we have indicated the regular subgroups of Aut(\(\Gamma_3\)) which contain a fixed 1-regular subgroup.

![Fig. 5](image)

Let A be an s-regular subgroup of Aut(\(\Gamma_3\)) and B a t-regular subgroup. Assume that \(A \cap B\) is 1-transitive. We claim that if \(s = 2\) or \(3\) and \(t = 4\) or \(5\) then \(A \cap B\) is 1-regular. This is clear since otherwise \(A \cap B\) would be k-regular with \(2 \leq k \leq 3\) which contradicts Theorem 3. Similarly, if \(s = t = 5\) and \(A \neq B\) then \(A \cap B\) is 1-regular. Otherwise \(A \cap B\) would be 4-regular and we would contradict Theorem 6 (ii).
11. Big Subgroups of $A_3$

Let $A_s = A_s^i$ (1 $\leq$ i $\leq$ 5) or $A_s^s$ (s = 2 or 4). A subgroup $K$ of $A_s$ is called big (in $A_s$) if $K \leq A_s$ and $K \cap (X_s \cup Y_s) \neq \{1\}$.

Proposition 19. We have $A_s^+ = X_s \ast yX_sy^{-1}$ and $A_s^+$ is the normal closure of $X_s$ in $A_s$.

Proof. The first assertion follows from [10, Theorem 6, p. 1 - 49]. The second is then obvious.

Proposition 20. The big subgroups of $A_1$ are $A_1^i$, $(A_1^i)^+$, and $E_1 = \langle y, ay^{-1}, a^{-1}ya \rangle$. We have $(A_1^1 : E_1) = 3$ and $E_1$ is the normal closure of $\langle y \rangle$.

Proof. If $K$ is a big subgroup of $A_1^i$ then either $K \ni X_1$ or $K \ni Y_1$ and all the claims follow easily from here.

Proposition 21. The big subgroups of $A_2$ are $A_2^i$, $(A_2^i)^+$, $E_2 = \langle ab, yaby \rangle$, $B' = \langle ab, y \rangle$, $B'' = \langle ab, yb \rangle$, and the normal closures of $\langle y \rangle$ and $\langle yb \rangle$ in $A_2$.

Proof. Let $K$ be a big subgroup of $A_2^i$. If $K \cap X_2 \neq \{1\}$ then $K \ni \langle ab \rangle$ and consequently $K = E_2$. Since $E_2 \triangleleft A_2^i$ and $A_2^i/E_2 \cong C_2 \times C_2$ it is clear that $K$ is one of $A_2^i, (A_2^i)^+, B', B''$, or $E_2$. Note that $E_2 = B' \cap B'' = B' \cap (A_2^i)^+ = B'' \cap (A_2^i)^+$. If $K \cap X_2 = \{1\}$ then $K \cap Y_2 = \langle y \rangle$ or $\langle yb \rangle$ and then $K$ must be the normal closure of $\langle y \rangle$ or $\langle yb \rangle$ in $A_2^i$.

Proposition 22. The big subgroups of $A_2 = A_2^e$ are $A_2^e, A_2^+$, and $E_2 = \langle ab, yaby^{-1} \rangle$. 
Proof. If \( K \triangleleft A_2 \) is big then \( U = K \cap X_2 \neq \{1\} \) because \( Y_2 = \langle y \rangle \cong C_4 \) and \( y^2 = b \in X_2 \). Since \( U \triangleleft X_2 \) we have \( U = \langle ab \rangle \) and \( K \supseteq E_2 \). It remains to note that \( A_2''/E_2 \cong C_4 \).

Proposition 23. The big subgroups of \( A_3 \) are \( A_3^1, (A_3^1)^+, A_2^1 = \langle ab, bc, y \rangle \), \( A_2' = \langle ab, bc, yb \rangle \), \( A_2' \cap A_2'' = (A_2^1)^+ = (A_2'')^+ \), and \( E_2 = \langle ac, yacy \rangle \).

Proof. Let \( K \) be a big subgroup of \( A_3^1 \). Since every non-trivial normal subgroup of \( Y_3 \cong D_4 \) contains the central element \( bc \) it follows that \( U = K \cap X_3 \neq \{1\} \). Since also \( U \triangleleft X_3 = \langle a, c \rangle \times \langle b \rangle \), \( \langle a, c \rangle \cong D_3 \), \( \langle b \rangle \cong C_2 \) we must have either \( ac \in U \) or \( b \in U \). Since \( a, b, c \) are all conjugate in \( A_3^1 \), we have in both cases \( ac \in U \) and so \( K \supseteq E_2 \). Our claim now follows from the fact that \( E_2 \triangleleft A_3^1 \) and \( A_3^1/E_2 \cong D_4 \).

Proposition 24. The big subgroups of \( A_4 = A_4^1 \) or \( A_4'' \) are \( A_4 \) and \( A_4^+ \).

Proof. Let \( K \) be a big subgroup of \( A_4 \). We must have again \( U = K \cap X_4 \neq \{1\} \).

We need only show that \( U = X_4 \). Assume the contrary. Then since \( U \triangleleft X_4 \cong S_4 \) we must have \( U = \langle b, c \rangle \). Since \( a, b, c, d \) are all conjugate in \( A_4 \) we obtain a contradiction.

Proposition 25. The big subgroups of \( A_5^1 \) are \( A_5^1, (A_5^1)^+, A_4^1 = \langle ab, bc, cd, de, y \rangle \), \( A_4'' = \langle ab, bc, cd, de, yc \rangle \), and \( A_4' \cap A_4'' = (A_4')^+ = (A_4'')^+ \).

Proof. Let \( K \) be a big subgroup of \( A_5^1 \). Since the center of \( Y_5 \) lies in \( X_5 \) we must have \( U = K \cap X_5 \neq \{1\} \). Recall that \( X_5 \cong S_4 \times C_2 \) and that \( \langle c \rangle \) is the center of \( X_5 \). It is easy to check that \( X_5 \) admits only two direct decompositions.
$X_5 = \langle ab, bc, cd, de \rangle \times \langle c \rangle,$
$X'_5 = \langle a, bc, cd, e \rangle \times \langle c \rangle.$

We cannot have $U \triangleleft \langle b, c, d \rangle$ because $a, b, c, d, e$ are all conjugate
in $A_5$. Since $U \triangleleft X_5$ it follows that $U$ is one of the following:
$X_5$, $\langle ab, bc, cd, de \rangle$, or $\langle a, bc, cd, e \rangle$. The last case is impossible since
$a, b, c, d, e$ are conjugate. Hence $U$ contains the normal closure of
$\langle ab, bc, cd, de \rangle$ in $A_5$ which coincides with $A_4^1 \cap A_4^2$. All the claims
now easily follow.
12. **Classification Problem**

An s'-object \((1 \leq s \leq 5)\) resp. an s"-object \((s = 2\) or \(4)\) is an ordered pair \((G,A)\) where \(G\) is a connected cubic graph and \(A\) is a subgroup of \(\text{Aut}(G)\) of type s' resp. s". An s-object is either an s'-object or an s"-object. We shall also need more special objects. For instance, a \((4',1')\) - object is an ordered triple \((G,A,B)\) where \(G\) is as above, \(A\) is a subgroup of \(\text{Aut}(G)\) of type \(4'\) and \(B\) is a subgroup of \(A\) of type \(1'\).

An s'-object \((G,A)\) is minimal if every covering morphism \((f,g):(G,A) \rightarrow (G',A')\), where \((G',A')\) is also an s'-object, is in fact an isomorphism, i.e., both \(f: G \rightarrow G'\) and \(g: A \rightarrow A'\) are isomorphisms. Minimal objects of other types are defined similarly.

If \(K\) is a small subgroup of \(A_s'\) resp. \(A_s''\) then we can associate to \(K\) an s'-object resp. s"-object \((G_K,A_K)\) as described in Theorem 1. Moreover every s'-object resp. s"-object is obtained in this way (up to isomorphism). We shall show in this section that if \((G,A)\) is an s'-object resp. s"-object then there is either one or two small subgroups \(K\) of \(A_s'\) resp. \(A_s''\) such that \((G_K,A_K) \cong (G,A)\). Hence the problem of classification of s'-objects resp. s"-objects reduces to the problem of classification of normal subgroups of \(A_s'\) resp. \(A_s''\).

Even in the simplest case, namely the case of 1'-objects, this last problem is very difficult and we do not expect that it will be solved soon. Indeed \(A_1'\) is the well-known modular group which has been studied for a long time (see section 14).
Lemma 4. Let $K_1, K_2$ be small subgroups of $A_s = A'_s$ or $A''_s$ such that $K_1 \leq K_2$. Then there is a natural covering morphism $(f,g): (G_{K_1}, A_{K_1}) \to (G_{K_2}, A_{K_2})$.

Proof. Define $g: A_{K_1} \to A_{K_2}$ to be the canonical map $A_s/K_1 \to A_s/K_2$, and define $f: G_{K_1} \to G_{K_2}$ to be the canonical map $A_s/\times K_1 \to A_s/\times K_2$. Then it is easy to check that $(f,g)$ is a covering morphism.

It is clear from this Lemma that an $s'$-object resp. $s''$-object $(G,K,A_K)$ is minimal if and only if $K$ is a maximal small subgroup of $A_s = A'_s$ resp. $A''_s$. One can use Zorn's lemma to show that every small subgroup of $A_s$ is contained in a maximal small subgroup of $A_s$. We shall say that a small subgroup $K$ of $A_s$ is even or odd according to whether $K \subset A^+_s$ or $K \not\subset A^+_s$.

We shall say that a graph $G$ is an $s'$-object or $s''$-object if $(G,\text{Aut}(G))$ is such an object.

If $(f,g): (G,A) \to (G',A')$ is a covering morphism we define $\ker(f,g) = \ker g$.

Lemma 5. Let $(f,g): (G,A) \to (G',A')$ be a covering morphism and $h \in A'_s$. Then $(hf,hg^{-1}): (G,A) \to (G',A')$ is also a covering morphism where $hgf^{-1}: A \to A'$ is defined by $(hgf^{-1})(a) = hg(a)f^{-1}$.

Proof. It is clear that $hf: G \to G'$ is a graph covering and that $hgf^{-1}: A \to A'$ is a group homomorphism. For $\alpha \in A$ we have $g(\alpha)f = f\alpha$ and consequently $(hgf^{-1})(\alpha)hgf = hg(\alpha)f^{-1}hf = hg(\alpha)f = h\alpha f$. Thus the lemma is proved.
Theorem 7. Let $(G,A)$ be an $s'$-object, $s = 3$ or 5. Then all covering morphisms $(f,g): (\Gamma_3,A_s) \to (G,A)$ have the same kernel.

Proof. Let $(f,g)$ and $(f',g')$ be two such covering morphisms. Further let $S$ be an $s$-arc in $\Gamma_3$. Now $f\circ S$ and $f'\circ S$ are two $s$-arcs in $G$. Since $A$ is $s$-transitive there exists an $h \in A$ such that $h \circ f\circ S = f'\circ S$.

By Lemma 5 $(hf,hgh^{-1})$ is also a covering morphism having the same kernel as $(f,g)$ and $hf$ and $f'$ coincide on $S(i)$, $0 \leq i \leq s$.

Hence without loss of generality we may assume that $f\circ S = f'\circ S$. It suffices to show that in this case $g = g'$.

If $x$ is a vertex of $\Gamma_3$ we let $\tilde{x} \in A_s$ be the involution defined in Proposition 3 or 5, and similarly for $y$ a vertex of $G$. It is clear from these Propositions that if $f(x) = y$ then $g(\tilde{x}) = \tilde{y}$.

Similarly, if $\alpha \in A$ is the unique involution such that $\alpha \circ S = S'$ is the opposite $s$-arc of $S$ then $g(\alpha) \in A$ is the unique involution such that $g(\alpha) \circ f\circ S$ is the opposite $s$-arc of $f\circ S$. Hence, it follows that $g(\tilde{x}) = g'(\tilde{x})$ for $x = S(i)$, $0 \leq i \leq s$ and $g(\alpha) = g'(\alpha)$. Since the elements $\tilde{x}$ for $x = S(i)$, $0 \leq i \leq s$ and $\alpha$ generate $A_s$, it follows that $g = g'$ and the proof is completed.

Lemma 6. Let $(f,g): (G,A) \to (G',A')$ be a covering morphism and let $h \in \text{Aut}(G)$ be in the normalizer of $A$. Then $(fh,g^h): (G,A) \to (G',A')$ is also a covering morphism, where $g^h(\alpha) = g(h\alpha h^{-1})$ for $\alpha \in A$.

Proof. This follows from $g^h(\alpha)fh = g(h\alpha h^{-1})fh = fh\alpha$.

Lemma 7. Let $(f,g): (G,A) \to (G',A')$ be a covering morphism and $A, A'$ be $s$-regular groups. Then $f(x) = f(y)$ iff there exists $\alpha \in \ker g$ such that $\alpha(x) = y$. 

Proof. Let \( y = \alpha(x) \) where \( \alpha \in \ker g \). Then
\[
f(y) = f(\alpha(x)) = (f \circ \alpha)(x) = (g(\alpha) \circ f)(x) = f(x).
\]
Conversely, let \( x \) and \( y \) be vertices of \( G \) such that
\( f(x) = f(y) \). We can choose \( s \)-arcs \( S_1 \) and \( S_2 \) such that \( S_1(0) = x \),
\( S_2(0) = y \), and \( f \circ S_1 = f \circ S_2 \). Since \( A \) is \( s \)-transitive there exists
\( \alpha \in A \) such that \( \alpha \circ S_1 = S_2 \). Then \( g(\alpha) \in A' \) fixes the \( s \)-arc \( S = f \circ S_1 \)
in \( G' \) and so \( g(\alpha) = 1 \) by \( s \)-regularity of \( A' \). Thus \( \alpha \in \ker g \).

Theorem 8. Let \( A_s = A'_s \), \( s = 1,2,4 \) or \( A_s = A''_s \), \( s = 2,4 \). Let \((G,A)\)
be an object of the same type as \((\Gamma_3,A'_s)\). Over all covering morphisms
from \((\Gamma_3,A'_s)\) to \((G,A)\) there are either one or two kernels and if there
is exactly one kernel then \( G \) is \((s + 1)\) - transitive.

Proof. Let \( S \) and \( T \) be fixed \( s \)-arcs in \( \Gamma_3 \) and \( G \), respectively. By
Lemma 5 we can consider only covering morphisms \((f,g)\): \((\Gamma_3,A'_s)\rightarrow(G,A)\)
such that \( f \circ S = T \).

Let \( S(0) = x \), \( T(0) = y \) and let \( (x,\alpha), (x,\beta), (y,\gamma), (y,\delta) \) be
shuntings such that \( \alpha,\beta \in A_s \); \( \gamma,\delta \in A \), \( \alpha \neq \beta \), \( \gamma \neq \delta \), \( \alpha(S(i)) = \beta(S(i)) = S(i+1), \gamma(S(i)) = \delta(S(i)) = S(i+1) \)
for \( 0 \leq i < s \). Then we must have
\[
\{g(\alpha), g(\beta)\} = \{\gamma, \delta\}
\]
and hence either \( g(\alpha) = \gamma \), \( g(\beta) = \delta \) or
\( g(\alpha) = \delta \), \( g(\beta) = \gamma \). Since \( A = \langle \alpha, \beta \rangle \) we conclude that there are at most
two possibilities for \( \ker g \). We know that at least one such covering
morphism exists (Theorem 1).

Assume now that there is only one such kernel, say \( \ker g = K \).

Note that \( A_s \trianglelefteq A'_{s+1} \) and in fact \( A'_{s+1} \) is the normalizer of \( A_s \) in
\( \text{Aut}(\Gamma_3) \) (Theorem 6). The uniqueness of \( K \) implies that \( K \trianglelefteq A'_{s+1} \)
and consequently \( G \) is \((s + 1)\) - transitive.
Proposition 26. Let \( (G,A) \) be a \( 1' \)-object, \( (u,v) \) an edge of \( G \), \( A(u) = \langle \rho \rangle \cong C_3 \) and \( A[u,v] = \langle \delta \rangle \cong C_2 \). The following are equivalent:

(i) There is a subgroup \( B \) of \( \text{Aut}(G) \) of type \( 2' \) containing \( A \);
(ii) There is an automorphism \( \phi \) of \( A \) such that \( \phi(\rho) = \rho^{-1} \), \( \phi(\sigma) = \sigma \).

Proof. (i) \( \Rightarrow \) (ii). Let \( \alpha \) be the generator of \( B(u,v) \cong C_2 \). Take \( \rho \) to be the restriction to \( A \) of the inner automorphism of \( B \) induced by \( \alpha \).

(ii) \( \Rightarrow \) (i). If \( \alpha, \beta \in A \) and \( \alpha(u) = \beta(u) \) then \( \alpha^{-1} \beta \in A(u) \) and \( \phi(\alpha^{-1} \beta) = \gamma \in A(u) \) so that \( \phi(\beta)(u) = \phi(\alpha) \gamma(u) = \phi(\alpha)(u) \). Therefore there exists a map \( f: G \to G \) such that \( f(\alpha(u)) = \phi(\alpha)(u) \) for all \( \alpha \in A \).

If \( \alpha \) is a vertex of \( G \) and \( \alpha = \alpha(u) \) then we have \( f(\beta(\alpha)) = f(\beta \alpha(u)) = f(\beta \phi(\alpha)(u)) = f(\beta)(f(\alpha)) \), i.e., \( f \circ \beta = f(\beta) \circ f \) for all \( \beta \in A \). It follows that \( f \) is surjective. It is also injective because \( f(\alpha(u)) = f(\beta(u)) \) implies \( \phi(\alpha)(u) = \phi(\beta)(u) \), \( \phi(\alpha^{-1} \beta) \in A(u) \) and consequently \( \alpha^{-1} \beta \in A(u) \), \( \alpha(u) = \beta(u) \).

If \( (a,b) \) is an edge of \( G \) then there exist \( \alpha \in A \) such that \( \alpha(u) = a \), \( \alpha(v) = b \) and hence \( f(a) = f(\alpha(u)) = \phi(\alpha)(u) \), \( f(b) = f(\alpha(v)) = \phi(\alpha)(v) \). Thus \( \{f(a), f(b)\} \) is also an edge of \( G \) and consequently \( f \in \text{Aut}(G) \).

We have also \( f^2 = 1 \) and \( f \circ \alpha \circ f^{-1} = \phi(\alpha) \) for all \( \alpha \in A \).

Since \( f \neq 1 \) and \( f(u) = u \), \( f(v) = f(y(u)) = \phi(y)(u) = y(u) = v \) we have \( f \notin A \). Hence \( B = \langle A, f \rangle \) is a 2-regular subgroup of \( \text{Aut}(G) \) and it must be of type \( 2' \) since \( B \supset A \).
Proposition 27. Let \((G,A)\) be a 2'-object, resp., 2''-object and 
a = \{u,v\}, \(b = \{v,w\}\) two distinct edges of \(G\). Let \(y \in A\) be an element 
which flips the edge \(b\). Then the following are equivalent.

(i) There is a subgroup \(B\) of \(\text{Aut}(G)\) of type 3' which 
contains \(A\);

(ii) There is an automorphism \(\phi\) of \(A\) such that \(\phi(a) = \tilde{a}\), 
\(\phi(b) = \tilde{b}\) and \(\phi(y) = y\tilde{b}\) resp., \(\phi(y) = y^{-1}\).

Proof. (i) \(\Rightarrow\) (ii). Let \(\alpha\) be the generator of \(B(u,v,w) \cong C_2\). Then we 
can take \(\phi\) to be the restriction to \(A\) of the inner automorphism of 
\(B\) induced by \(\alpha\).

(ii) \(\Rightarrow\) (i). There is a unique map \(f: G \to G\) such that 
\(f(\alpha(v)) = \phi(\alpha)(v)\) for all \(\alpha \in A\). This \(f\) satisfies 
\(f \circ \alpha = \phi(\alpha) \circ f\) for all \(\alpha \in A\). It is easy to check that \(f \in \text{Aut}(G)\), \(f^2 = 1\), \(f \neq 1\). 
Since \(f\) fixes \(u,v,w\) we have \(f \not\in A\) and hence \(B = \langle A, f \rangle\) is 
3-regular.

Proposition 28. Let \((G,A)\) be a 4'-object, resp., 4''-object and let 
a = \{v_0,v_1\}, \(b = \{v_1,v_2\}\), \(c = \{v_2,v_3\}\), \(d = \{v_3,v_4\}\) be distinct edges 
of \(G\). Let \(y \in A\) be an element of order 2 resp. 4 such that \(y(v_1) = v_4\), 
\(y(v_2) = v_3\) and \(y(v_4) = v_1\). The following are equivalent:

(i) There is a subgroup \(B\) of \(\text{Aut}(G)\) of type 5' containing 
\(A\);

(ii) There is an automorphism \(\phi\) of \(A\) such that \(\phi(\tilde{a}) = \tilde{a}\), 
\(\phi(\tilde{b}) = \tilde{b}\), \(\phi(\tilde{c}) = \tilde{c}\), \(\phi(\tilde{d}) = \tilde{d}\) and \(\phi(y) = y\tilde{c}\) resp., \(\phi(y) = y^{-1}\).
Proof. (i) ⇒ (ii). Let \( \alpha \) be the generator of \( B(v_0, v_2, v_4) \cong C_2 \). Then we can take \( \phi \) to be the restriction to \( A \) of the inner automorphism of \( B \) induced by \( \alpha \).

(ii) ⇒ (i). There is a unique map \( f: G \rightarrow G \) such that \( f(\alpha(v_2)) = \phi(\alpha)(v_2) \) holds for all \( \alpha \in A \). It follows that \( f \circ \alpha = \phi(\alpha) \circ f \) for all \( \alpha \in A \). Again, it is easy to verify that \( f \) is an automorphism of \( G \) of order 2. Since \( f \) fixes the vertices \( v_i, 0 \leq i \leq 4 \) and \( f \neq 1 \) it follows that \( f \notin A \). Thus \( B = \langle A, f \rangle \) is 5-regular.
13. Finite Primitive Objects

An $s$-object $(G,A)$ is said to be primitive if $A$ acts primitively on the vertex-set of $G$. A graph $G$ is called vertex-primitive if $\text{Aut}(G)$ acts primitively on the vertex set of $G$.

We shall list here all finite primitive $s$-objects ($1 \leq s \leq 5$).

Note that if $(G,A)$ is a finite primitive, say, $2'$-object then $G$ is a vertex-primitive graph. Since all finite 3-valent vertex-primitive graphs are known [16] it is easy to deduce the following facts.

**Proposition 29.** There is only one finite primitive $1'$-object: $(K_4,A_4)$ where $A_4$ stands for the alternating group on four letters.

There are precisely two finite primitive $2'$-objects: $(K_4,S_4)$ and $(P,A_5)$, where $P$ is the Petersen graph and $A_5$ the alternating group.

There is only one finite primitive $2''$-object: $(G(28),\text{PSL}_2(7))$.

There are precisely two finite primitive $3'$-objects: $(P,S_5)$ and $(G(28),\text{PGL}_2(7))$.

The finite primitive $4'$-objects form an infinite series $(G_p,\text{PSL}_2(p))$ where $p$ is a prime congruent to $\pm 1 \pmod{16}$.

There is only one finite primitive $4''$-object: $(G(234),\text{SL}_3(3))$.

There is only one finite primitive $5'$-object: $(G(234),\text{Aut}(\text{SL}_3(3)))$.

For the definition of the graphs $G(28), G(234)$, and $G_p$ we refer to [2], [1, p. 125], and [16].
14. Connection with the Modular Group

The group $\text{PSL}_2(Z)$ is the well-known modular group. It consists of 2 by 2 integral matrices of determinant 1 in which one identifies every matrix with its negative, i.e., $\text{PSL}_2(Z) = \text{SL}_2(Z)/\{\pm I\}$. It is well-known that $\text{PSL}_2(Z) \cong \mathbb{C}_3 \ast \mathbb{C}_2$ see [9, p. 139].

Proposition 30. We have $A_1' \cong \text{PSL}_2(Z)$, $A_2' \cong \text{PGL}_2(Z)$, and $A_3' \cong \text{Aut PGL}_2(Z)$.

Proof. The first claim follows from the above remark and the definition of $A_1'$.

Let $a' = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $b' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ considered as elements of $\text{PGL}_2(Z)$. Then it is easy to check that the amalgam $(<a',b'>, <b',y'>)$ is isomorphic to the Amalgam $2'$. Hence there exists a homomorphism $f: A_2' \rightarrow \text{PGL}_2(Z)$ such that $f(a) = a'$, $f(b) = b'$, and $f(y) = y'$. Since $\text{PGL}_2(Z)$ is generated by $a', b', y'$, the homomorphism $f$ is surjective. The subgroup $<ab, y> = <ab> \ast <y> \cong \mathbb{C}_3 \ast \mathbb{C}_2$ of $A_2'$ is mapped by $f$ onto $\text{PSL}_2(Z)$. Since $f(ab) = a'b'$, $f(y) = y'$ and $\text{PSL}_2(Z) = <a'b'> \ast <y'>$ it follows that $\ker f \cap <ab, y> = \{1\}$. Since the index of $<ab, y>$ in $A_2'$ is 2 and $<ab, y>$ is not a direct factor of $A_2'$ it follows that $f$ is injective and consequently an isomorphism.

There is an automorphism $\alpha$ of $A_2'$ such that $\alpha(a) = a$, $\alpha(b) = b$, $\alpha(y) = by$. We have $\alpha^2 = 1$ and we claim that $\text{Aut}(A_2') = A_2' \ast <\alpha>$. Since $A_2'$ has trivial center we are allowed to consider $A_2'$ as a normal subgroup of $\text{Aut}(A_2')$. Our claim will follow if we prove that $\text{Aut}(A_2') = A_2' \cup \alpha \circ A_2'$. Let $\beta \in \text{Aut}(A_2')$. Then $\beta(X_2) \cong X_2 \cong D_3$ and by Kurosh subgroup theorem there exists an inner automorphism $\gamma$ such that $\gamma \beta(X_2) = X_2$. 

Since $X_2 \cong D_3$ and all three involutions of $D_3$ are conjugate to each other, there exists an inner automorphism $\delta$ of $A_2$ such that
\[
\delta \gamma \beta (X_2) = X_2 \quad \text{and} \quad \delta \gamma \beta (b) = b.
\]
Then $\delta \gamma \beta (a)$ is either $a$ or $aba = bab$.

In any case there is an inner automorphism $\epsilon$ of $A_2$ such that $\epsilon (X_2) = X_2$, $\epsilon (b) = b$ and $\epsilon \delta \gamma \beta (a) = a$. Thus $\epsilon \delta \gamma \beta$ fixes every element of $X_2$.

Since $Y_2$ is the centralizer of $b$, we have $\epsilon \delta \gamma \beta (Y_2) = Y_2$. Thus either $\epsilon \delta \gamma \beta (y) = y$ or $\epsilon \delta \gamma \beta (y) = yb$. In the former case we have $\epsilon \delta \gamma \beta = 1$ and in the latter $\epsilon \delta \gamma \beta = \alpha$. Hence our claim about $\text{Aut}(A_2)$ is proved.

Since $A_2^1 \leq A_3^1$ and $(A_3^1:A_2^1) = 2$ we have a homomorphism $g: A_3^1 \to \text{Aut}(A_2^1)$ induced by conjugation. Since the centralizer of $A_2^1$ in $A_3^1$ is trivial, this is an embedding. Since $\text{Aut}(A_2^1) = A_2^1 \rtimes \langle \alpha \rangle$, $g$ is an isomorphism.
15. Some Subgroups of $A_3$

Some subgroups of $A_3$ are shown on Fig. 6 and we shall describe them now. The numbers on this figure are the indices of various subgroups. $B'$ and $B''$ are the two 1-regular subgroups of $A_2$, $E_3 = (A_2)^+$ and $E_2 = B' \cap B''$. The subgroup $E'$ (resp. $E''$) is the unique normal subgroup of $B'$ (resp. $B''$) of index 3. We know that $B'$ and $B''$ are conjugate in $A_3$ and so are $E'$ and $E''$. Note that $A_2$ has no normal subgroups of index 3 because every homomorphism $\chi_2 \rightarrow C_3$ or $\psi_2 \rightarrow C_3$ is trivial. Hence $A_2/E' \cong A_2/E'' \cong D_3$ and $E'E'' \cong A_2$. If we put $T = E' \cap E''$ then $E'/T \cong E''/T \cong D_3$. The group $U' = E' \cap E_2$ (resp. $U'' = E'' \cap E_2$) is the commutator subgroup of $B'$ (resp. $B''$) and $B'/U' \cong B''/U'' \cong C_6$ [9, p. 141].

The group $E_2/T$ is elementary abelian of order 9 and $A_3$ acts on it as a cyclic group of order 2. Therefore there are precisely two subgroups $V, W$ such that $E_2 \supseteq V \supseteq T$, $E_2 \supseteq W \supseteq T$, $V/T \cong W/T \cong C_3$, $V \triangleleft A_3$, and $W \triangleleft A_3$. The subgroups $U', U'', V, W$ are normal in $A_2$ but they are neither big nor small in $A_2$. The $3'$-object which is associated to $T \triangleleft A_3$ is bipartite and has 6 vertices. It is easy to see that this object must be $(K_3, 3, \text{Aut}(K_3, 3))$.

$K'_4$ and $K''_4$ are subgroups which correspond to the 2-object $(K_4, S_4)$. Next we put $Q' = K'_4 \cap E_2$, and $Q'' = K''_4 \cap E_2$. Then $Q'$ and $Q''$ are the normal subgroups of $A_2$ which correspond to the graph of the cube. The product $L = K'_4K''_4$ gives rise to the normal subgroup $L/K'_4$ of $A_2/K'_4 \cong S_4$. If $L \neq A_2$ then we would have $L \subset B'$ since $B'/K'_4 \cong A_4$ and every proper normal subgroup of $S_4$ is contained in $A_4$ (the alternating
Fig. 6
But L \subseteq B' is impossible since K''_4 \nsubseteq B'. Thus we have K'_4K''_4 = A'_2, and if N = K'_4 \cap K''_4 then K'_4/N \cong K''_4/N \cong S_4. We also have Q'/N \cong Q''/N \cong A_4 (the alternating group).

We have T/N \cong (T\cap Q'/N) \times ((T\cap Q'')/N) and so T/N is elementary abelian group of order 16. Moreover, we have A''_2/N \cong (K''_4/N) \times (K''_4/N) \cong S_4 \times S_4.

It follows that T/N is simple as an A_3^'-module, i.e., there are no normal subgroups of A_3^' which lie strictly between N and T. The graph corresponding to N \triangleleft A_3^' is 3-regular; it is a 12-fold cover of the cube and a 16-fold cover of K_{3,3}'.

The subgroup P \triangleleft A_3^' corresponds to the Petersen graph; D' and D'' are the two normal subgroups of A_2^' which correspond to the dodecahedron, and Z corresponds to Desargues' graph which is 3-regular.

Finally K \triangleleft A_3^' corresponds to the vertex primitive 3-regular graph \( G(28) \).

Preposition 31. Let \((G,A)\) be a 3'-object. 1) If \( A \) contains a 1-regular subgroup then \( G \) is bipartite and \( A \) contains two 2-regular subgroups and two 1-regular subgroups. 2) If \( A \) contains two 2-regular subgroups then \( G \) is bipartite.

Proof. Let \((f,g): (I_3^',A_3^') \rightarrow (G,A)\) be a covering morphism. The inverse image in \( A_3^' \) of a 1-regular (resp. 2-regular) subgroup of \( A \) is \( B' \) or \( B'' \) (resp. \( A'_2 \) or \( A''_2 \)). Thus if \( K = \ker g \) then \( K \subseteq B' \) or \( K \subseteq B'' \) (resp. \( K \subseteq A'_2 \cap A''_2 \)). If \( K \subseteq B' \) then also \( K \subseteq B'' \) since \( B' \) and \( B'' \) are conjugate in \( A_3^' \). Thus it follows that \( K \subseteq E_2 \) (resp. \( E_3 \)) and so \( G \) is bipartite by Proposition 17. For the case \( K \subseteq E_2 \), we have \( K \subseteq B',B'', A'_2 \), and \( A''_2 \). Thus, by Theorem 1, the images of these four subgroups under \( g \) act 1-regular (resp. 2-regular) in \( G \).
16. Some Infinite Classes of 2-regular Cubic Graphs

In this section \( p \) will denote a prime \( \geq 5 \). All the matrices stand for the corresponding elements of \( \text{PGL}_2(p) \).

Let
\[
\begin{align*}
a &= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, & b &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & y &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{align*}
\]

Since
\[
aby = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \quad yab = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix},
\]
it follows from a theorem of Dickson [6, p. 44] that \( \langle ab, y \rangle = \text{PSL}_2(p) \).

Note that we have
\[
a^2 = b^2 = y^2 = (ab)^2 = (by)^2 = (ab)^3 = 1.
\]

We define the graph \( G_p \) as follows: its vertices are the cosets \( u < ab >, \ u \in \text{PSL}_2(p) \) and the edges are \( \{ u < ab >, v < ab > \} \) for \( u, v \in \text{PSL}_2(p) \) and \( u^{-1}v \in < ab > y < ab > \).

**Proposition 3.** The graph \( G_p \) is a 2-regular cubic graph of odd girth.
We have \( \text{Aut}(G_p) \cong C_2 \times \text{PSL}_2(p) \) or \( \text{PGL}_2(p) \) according to whether \( -1 \) is a square or not mod \( p \).

**Proof.** It is clear that \( (G_p, \text{PSL}_2(p)) \) is a 1'-object. Since \( \text{PSL}_2(p) \) is simple, it has no subgroups of index 2. Hence \( G_p \) is not bipartite.

\( \text{PSL}_2(p) = \langle ab, y \rangle \) admits an automorphism \( \phi \) such that \( \phi(ab) = ba = (ab)^{-1} \) and \( \phi(y) = y \). Indeed \( \phi \) is the restriction to \( \text{PSL}_2(p) \) of the inner automorphism of \( \text{PGL}_2(p) \) induced by \( b \). We have \( b \in \text{PSL}_2(p) \) precisely when \( -1 \) is a square mod \( p \).
From Proposition 26 and its proof it follows that \( \text{Aut}(G'_p) \) has a subgroup \( A \) which contains \( \text{PSL}_2(p) \) and \( A \cong C_2 \times \text{PSL}_2(p) \) or \( A \cong \text{PGL}_2(p) \) according to whether \(-1\) is a square or not modulo \( p \).

In both cases \( \text{PSL}_2(p) \triangleleft \text{Aut}(G'_p) \) and so Proposition 14 implies that \( A = \text{Aut}(G'_p) \).

Now let \( A = \langle a, b, y \rangle \). Since \( \langle ab, y \rangle = \text{PSL}_2(p) \) we have either \( A = \text{PSL}_2(p) \) or \( A = \text{PGL}_2(p) \). The first case occurs if \(-1\) is a square modulo \( p \), i.e., if \( p \equiv 1 \pmod{4} \), and the second case otherwise.

We define a graph \( G_p'' \) as follows: its vertices are the cosets \( u \langle a, b \rangle, u \in A \) and its edges are \( \{u \langle a, b \rangle, v \langle a, b \rangle\} \) for \( u, v \in A \) and \( u^{-1}v \in \langle a, b \rangle \). Since the amalgam \( \langle a, b \rangle, \langle b, y \rangle \) is isomorphic to \( (X_2, Y_2') \) it is clear that \( (G_p'', A) \) is a 2'-object.

**Proposition 33.** The graph \( G_p'' \) is cubic and has odd girth. If \( p > 5 \) then \( G_p'' \) is 2-regular and \( \text{Aut}(G_p'') = A \), while \( G_5'' \) is the Petersen graph.

**Proof.** If \( A = \text{PSL}_2(p) \) then \( G_p'' \) is not bipartite because \( A \) has no subgroups of index 2. If \( A = \text{PGL}_2(p) \) then \( G_p'' \) is not bipartite because \( A \) has only one subgroup of index 2, namely \( \text{PSL}_2(p) \) and \( \langle a, b \rangle \notin \text{PSL}_2(p) \).

Thus in both cases \( G_p'' \) has odd girth.

If \( A = \text{PGL}_2(p) \) then the subgroup \( \text{PSL}_2(p) \) is 1-regular. Since every \( (3', 1') \)-object is bipartite (Proposition 31) it follows that \( G_p'' \) is not 3-regular. Hence by Theorem 3, \( G_p'' \) is 2-regular in this case.

Now assume that \( A = \text{PSL}_2(p) \), i.e., \( p \equiv 1 \pmod{4} \). Thus \( a, b \in \text{PSL}_2(p) \).
By Proposition 26 we have to show that $\text{PSL}_2(p)$ has no automorphisms $\phi$ such that $\phi(a) = a$, $\phi(b) = b$, and $\phi(y) = y$ when $p > 5$.

It is known [3, Chapter 4] that the automorphism group of $\text{PSL}_2(p)$ is isomorphic to $\text{PGL}_2(p)$. In fact every element of $\text{PGL}_2(p)$ induces by conjugation an automorphism of $\text{PSL}_2(p)$ and this gives us an isomorphism $\text{PGL}_2(p) \rightarrow \text{Aut(PSL}_2(p))$. Assume that $\phi$ exists and is induced by $z = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Since $zb = bz$ we find that

$$\lambda \begin{pmatrix} \beta & \alpha \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix}$$

for some $\lambda$. Thus $\gamma = \lambda \beta$, $\beta = \lambda \gamma$, $(\lambda^2 - 1)\beta = 0$. We cannot have $\beta = 0$ since then also $\gamma = 0$ and $zy \neq byz$. Therefore $\beta \neq 0$ and $\lambda = \pm 1$.

Thus

$$z = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \text{ or } z = \begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix}.$$ 

If $z = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$ then $zy = byz$ gives

$$\mu \begin{pmatrix} -\beta & \alpha \\ -\alpha & \beta \end{pmatrix} = \begin{pmatrix} -\alpha & -\beta \\ -\beta & -\alpha \end{pmatrix}$$

for some $\mu$. Hence $\alpha = \mu \beta$, $\beta = -\mu \alpha$ and so $\mu^2 = -1$ and $z = \begin{pmatrix} 1 & \mu \\ \mu & 1 \end{pmatrix}$.

The equation $za = az$ now forces $2\mu = -1$. Thus $1 = 4\mu^2 = -4 \pmod{p}$ and so $p = 5$.

If $z = \begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix}$ then $zy = byz$ gives

$$\mu \begin{pmatrix} -\beta & \alpha \\ \alpha & -\beta \end{pmatrix} = \begin{pmatrix} -\alpha & -\beta \\ -\beta & -\alpha \end{pmatrix}$$
for some $\mu$. Again $\alpha = \overline{\alpha} \beta$, $\beta = -\mu \alpha$, $\mu^2 = -1$ and so $z = (\begin{smallmatrix} 1 & -\mu \\ \mu & -1 \end{smallmatrix})$.

The equation $za = az$ now forces $\mu = 0$ which is a contradiction.

We leave to the reader to verify that in case when $-1$ is a square modulo $p$ then we have a canonical covering map $G'_p \to G''_p$ sending $u<ab>$ to $u<a,b>$ for $u \in \text{PSL}_2(p)$. This is a two-fold covering and it is compatible with $\text{Aut}(G'_p)$. 
17. Some Subgroups of \( A_5 \)

Some subgroups of \( A_5 \) are shown on Fig. 7 and we shall describe them now. We have \( E_5 = A_4 \cap A_4 = (A_4)^+ \). \( T \) is the normal subgroup of \( A_5 \) which corresponds to Tutte's 8-cage. As is well-known we have \( E_5/T \cong A_6 \), \( (A_5)^+/T \cong S_6 \), and \( A_5/T \cong \text{Aut}(S_6) \).

\( M \) is the normal subgroup of \( A_5 \) which corresponds to the unique finite primitive 5'-object \( (G(234), \text{Aut}(SL_3(3))) \). We have \( A_4/M \cong SL_3(3) \) and \( A_5/M \cong \text{Aut}(SL_3(3)) \).

For each prime \( p \equiv \pm 1 \pmod{16} \) we denote by \( N_p^1 \) and \( N_p^\prime \) two normal subgroups of \( A_4 \) which correspond to the primitive 4'-object \( (G_p, PSL_2(p)) \).

By \( B' \) and \( B'' \) we denote two 1-regular subgroups of \( A_4 \) which are not conjugate in \( A_4 \). Each of \( B' \) and \( B'' \) has eight conjugates in \( A_4 \) and of course \( B' \) and \( B'' \) are conjugate in \( A_5 \) (Theorem 4).

Let \( H' \), resp. \( H'' \) be the intersection of the conjugates of \( B' \) resp. \( B'' \).

Finally we put \( K = H' \cap H'' \).

Proposition 34. The Heawood's graph is a unique minimal \( (4', 1') \)-object.

We have \( H' \leq (A_4)^+ \) and so every \( (4', 1') \)-object is a covering of Heawood's graph. Further, \( (B': H') = 42 \), \( (A_4)^+/H' \cong PSL_2(7) \), and \( A_4/H' \cong PGL_2(7) \).

Proof. Let \( \Pi \) be the projective plane over the Galois field \( GF(2) \). The vertices of the Heawood's graph \( H \) are the points and lines of \( \Pi \). A point-vertex is joined by an edge to a line-vertex if and only if this point and line are incident in \( \Pi \). Thus \( H \) is a connected, bipartite, cubic graph of girth 6, having 14 vertices. \( H \) is 4-regular, \( \text{Aut}(H) \cong PGL_2(7) \) and
Fig. 7
Aut(H) has 1-regular subgroups. Thus H is a \((4',1')\)-object. Since 
\(\text{PSL}_2(7)\) acts primitively on point-vertices as well as on line-vertices, 
it follows that H is minimal as a \((4',1')\)-object.

Now, let \((G,A,B)\) be a \((4',1')\)-object, thus \(B \leq A \leq \text{Aut}(G)\),
A is 4-regular, and B is 1-regular. Let \(f: \Gamma_3 \to G\) be any covering.
Then we can lift the groups A and B to \(\tilde{\Gamma}_3\). More precisely, we define
\(\tilde{A}\) to be the subgroup of \(\text{Aut}(\tilde{\Gamma}_3)\) consisting of all automorphisms \(\phi\)
of \(\tilde{\Gamma}_3\) for which there exists an automorphism \(a \in A\) such that \(a \circ \phi = f \circ \phi\).
One defines \(\tilde{B}\) similarly. Then \(\tilde{A}\) is 4-regular and \(\tilde{B}\) is 1-regular
(see [4]). Hence we may assume that \(\tilde{A} = A_4'\) and \(\tilde{B} = B'\), say. Let \(g: \tilde{A} \to A\)
be the homomorphism induced by the covering \(f: \Gamma_3 \to G\). Then \(\ker g = K \triangleleft A_4'\)
and \(K \leq B'\).

This implies that \(K \leq H'\).

Hence the \((4',1')\)-object associated by Theorem 1 to the small
subgroup \(H'\) of \(A_4'\) is a unique minimal \((4',1')\)-object. Thus this
object must be H. Since the Heawood's graph is bipartite we must have
\(H' \leq (A_4')^+\). All other assertions follow from the properties of Heawood's graph.

**Proposition 35.** There is a unique minimal \((5',1')\)-object; this is the object
\((G_p,A_p)\) associated to the small subgroup \(P = H' \cap H''\) of \(A_5'\). We have
\(P \leq (A_5')^+\) and so every \((5',1')\)-object has the following properties:

1) is a covering of Heawood's graph;
2) is a covering of \(G_p\);
3) contains a 4'-regular subgroup;
4) is bipartite.

**Proof.** We omit it because it is very similar to the proof of the previous
Proposition.
18. **Embedding a 1'-object into an orientable surface.**

Let \((G, A)\) be a 1'-object. Here we shall consider the graph \(G\) realized as a topological space in the usual way, see [10, p. 1-20].

We know that there are precisely two \(A\)-conjugacy classes of \(A\)-shuntings (Lemma 2); we fix one of these classes and refer to it and the shuntings belonging to it as positive. If \((a, \alpha)\) is an \(A\)-shunting then its trajectory is either a circuit or a doubly infinite path in \(G\). If \(\{u, v\}\) is an edge belonging to this trajectory and \(\alpha(u) = v\) then we assign to this edge the orientation from \(u\) to \(v\). This oriented trajectory will be called an \(A\)-trajectory, it is positive if the corresponding shunting \((a, \alpha)\) is positive.

Now let \((a, b)\) be an edge of \(G\). Let \((a, \alpha)\) and \((b, \beta)\) be positive \(A\)-shuntings such that \(\alpha(a) = b\) and \(\beta(b) = a\). Then Proposition 10 implies that \(\beta(a) \neq \alpha^{-1}(a)\) and so \((a, \alpha)\) and \((b, \beta)\) have different trajectories. Hence it follows that every edge of \(G\) belongs to precisely two positive \(A\)-trajectories and the orientations assigned to it in these \(A\)-trajectories are opposite. Thus if we glue a 2-cell along each positive \(A\)-trajectory then we shall obtain an orientable surface.

All positive \(A\)-trajectories have the same length, say \(n\); if these trajectories are doubly infinite paths then \(n = \infty\). If \((a, \alpha)\) is a positive \(A\)-shunting then \(n\) is the order of \(\alpha\).

**Proposition 36.** Let \((G, A)\) be a 1'-object, \((a, \alpha)\) an \(A\)-shunting and \(n\) the order of \(\alpha\). If \(G\) is finite with \(v\) vertices then there is an embedding of \(G\) into an orientable closed surface \(S\) of genus
Moreover, the action of $A$ on $G$ extends to an action of $A$ on $S$.

Proof. All the claims follow from the preceding discussion except the formula for the genus. Let $e$ be the number of edges of $G$ and $f$ the number of 2-cells used in constructing $S$. Each of these 2-cells together with its boundary is an $n$-gon. Hence we have $nf = 3v = 2e$. The Euler characteristic of $S$ is

$$\chi = v - e + f = \frac{6-n}{2n} v.$$ 

since $2g = 2 - \chi$ we are done.

For instance, $(K_4, A_4)$ is a 1'-object. The $A_4$-shuntings have order 3 and we obtain $g = 0$. Thus we obtain the ordinary embedding of $K_4$ into the sphere.

Since $g$ is an integer, this Proposition can be used to show that certain graphs do not have 1-regular groups of automorphisms.
19. **Some Open Questions**

We first give an example of a 3-regular graph which has no 2-regular group of automorphisms.

Let
\[
\begin{align*}
    a &= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \\
    b &= \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, \\
    c &= \begin{pmatrix} 9 & 1 \\ 1 & 0 \end{pmatrix}, \\
    y &= \begin{pmatrix} 4 & -3 \\ 3 & -4 \end{pmatrix}
\end{align*}
\]
be in \( \text{PGL}_2(13) \). Then in fact \( a, b, c \) are in \( \text{PSL}_2(13) \) but \( y \) is not.

Since
\[
    ay = \begin{pmatrix} -2 & 2 \\ -1 & -3 \end{pmatrix}
\]
has order 14 and
\[
    (ya)^2(yay)^3 = \begin{pmatrix} 11 & 0 \\ -3 & 1 \end{pmatrix}
\]
has order 13 it is clear that the elements \( a, b, c, y \) generate \( \text{PGL}_2(13) \).

These elements satisfy
\[
\begin{align*}
    a^2 &= b^2 = c^2 = y^2 \\
    (ab)^2 &= (bc)^2 = (cy)^2 = (ac)^3 = 1, \\
    yby &= bc
\end{align*}
\]

Hence the amalgam \( \langle a, b, c, y \rangle \) is isomorphic to \( \langle X_3, Y_3 \rangle \). We define a graph \( G \) as follows: its vertices are the cosets \( u<a, b, c> \) for \( u \in \text{PGL}_2(13) \) and its edges are \( \{u<a, b, c>, v<a, b, c>\} \) for \( u, v \in \text{PGL}_2(13), u^{-1}v \in a, b, c \rangle \). Then \( \text{PGL}_2(13) = \text{Aut}(G) \) and \( G \) is a 3'-object. Since \( |\text{PGL}_2(13)| = 13 \cdot 168 \) and \( |<a, b, c>| = 12 \) it follows that \( G \) has 13 \cdot 14 = 182 \) vertices. Since \( \text{PSL}_2(13) \) is simple it is clear that it is the unique subgroup of \( \text{PGL}_2(13) \) of index 2. The graph \( G \) is bipartite because \( <a, b, c> \leq \text{PSL}_2(13) \) and \( \text{Aut}(G) \) is not generated by the vertex-fixers. Consequently \( G \) has no 2-regular group of automorphisms.
We shall now list a few questions whose answers still evade us.

**Problem 1.** Construct a finite graph $G$ which is 5-regular and has no 4-regular groups of automorphisms.

**Problem 2.** Is there a finite connected cubic graph $G$ such that $\text{Aut}(G)$ is 2-regular of type $2^1$?

**Problem 3.** Is there a finite connected cubic graph $G$ such that $\text{Aut}(G)$ is 4-regular of type $4^3$?

**Problem 4.** Is every minimal $s'$ or $s''$-object finite?

**Problem 5.** Find all simple groups $A$ (finite or infinite) such that $A$ is generated by an element of order 2 and an element of order 3. (This is in fact a well-known problem of finding all simple quotients of the modular group.)

**Problem 6.** Let $A$ be an $s$-regular subgroup of $\text{Aut}(\Gamma_3)$ and $B$ a $t$-regular subgroup of $\text{Aut}(\Gamma_3)$. Assume that $A \cap B$ is a 1-regular group. Is it true that the subgroup $\langle A, B \rangle$ of $\text{Aut}(\Gamma_3)$ is a free product with amalgamation of $A$ and $B$ with $A \cap B$ amalgamated?

**Problem 7.** Let $A$ and $B$ be two subgroups of $\text{Aut}(\Gamma_3)$ of type $4'$ such that $A \cap B$ is 1-regular. The subgroup $D = \langle A, B \rangle$ of $\text{Aut}(\Gamma_3)$ is $\omega$-transitive. Is it true that the even subgroup $D^+$ of $D$ is simple?

**Problem 8.** Determine the girth of the graphs $G_p$ (see Section 13). (It is known [2] that the girth of $G_{17}$ is 9.)
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