Llull and Copeland Voting Computationally Resist Bribery and Control

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Abstract

Control and bribery are settings in which an external agent seeks to influence the outcome of an election. Constructive control of elections refers to attempts by an agent to, via such actions as addition/deletion/partition of candidates or voters, ensure that a given candidate wins [BTT92]. Destructive control refers to attempts by an agent to, via the same actions, preclude a given candidate’s victory [HHR07a]. An election system in which an agent can affect the result and in which recognizing the inputs on which the agent can succeed is NP-hard (polynomial-time solvable) is said to be resistant (vulnerable) to the given type of control. Aside from election systems with an NP-hard winner problem, the only systems previously known to be resistant to all the standard control types are highly artificial election systems created by hybridization [HHR07b].

We study a parameterized version of Copeland voting, denoted by Copeland$^\alpha$, where the parameter $\alpha$ is a rational number between 0 and 1 that specifies how ties are valued in the pairwise comparisons of candidates. In every previously studied constructive or destructive control scenario, we determine which of resistance or vulnerability holds for Copeland$^\alpha$ for each rational $\alpha$, $0 \leq \alpha \leq 1$. In particular, we prove that Copeland$^{0.5}$, the system commonly referred to as “Copeland voting,” provides full resistance to constructive control. Among the systems with a polynomial-time winner problem, this is the first natural election system proven to have full resistance to constructive control. In addition, we prove that both Copeland$^0$ and Copeland$^1$ (interestingly, the latter is an election system developed by the thirteenth-century mystic Ramon Llull) are resistant to all the standard types of constructive control other than one variant of addition of candidates. Moreover, we show that for each rational $\alpha$, $0 \leq \alpha \leq 1$, Copeland$^\alpha$ voting is fully resistant to bribery attacks, and we establish fixed-parameter tractability of bounded-case control for Copeland$^\alpha$.

We also study Copeland$^\alpha$ elections under more flexible models such as microbribery and extended control, we integrate the potential irrationality of voter preferences into many of our results, and we prove our results in both the unique-winner and the nonunique-winner model. Our vulnerability results for microbribery are proven via a technique involving min-cost network flow.
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1 Introduction

1.1 Some Historical Remarks: Llull’s and Copeland’s Election Systems

Elections have played an important role in human societies for thousands of years. For example, elections were of central importance in the democracy of ancient Athens. There citizens typically could only agree (vote yes) or disagree (vote no) with the speaker, and simple majority-rule was in effect. The mathematical study of elections, give or take a few discussions by the ancient Greeks and Romans, was until recently thought to have been initiated only few of hundred years ago, namely in the breakthrough work of Borda and Condorcet—later in part reinvented by Dodgson (see, e.g., [MU95] for reprints of their classic papers). One of the most interesting results of this early work is Condorcet’s observation [Con85] that if one conducts elections with more than two alternatives then even if all voters have rational (i.e., transitive) preferences, the society in aggregate can be irrational (indeed, can have cycles of strict preference). Based on his observations, Condorcet suggested that if there exists a candidate $c$ such that $c$ defeats any other candidate in a head-to-head contest then that candidate should win the election. Such a candidate is called a Condorcet winner. Clearly, there can be at most one Condorcet winner in any election and there might be none.

This understanding of history has been reconsidered during the past few decades, as it has been rediscovered that the study of elections was in fact considered deeply as early as the thirteenth century (see Hägeler and Pukelsheim [HP01] and the citations therein regarding Ramon Llull and the fifteenth-century figure Cusanus, especially the citations that there are numbered 3, 5, and 24–27). Ramon Llull (b. 1232, d. 1315), a Catalan mystic, missionary, and philosopher developed an election system that (a) has an efficient winner-determination procedure and (b) elects a Condorcet winner whenever one exists and otherwise elects candidates that are, in some sense, closest to being Condorcet winners.

Llull’s motivation for developing an election system was to obtain a method of choosing the abbesses, abbots, bishops, and perhaps even the pope. His election ideas never gained public acceptance in medieval Europe and were long forgotten.

It is interesting to note that Llull allowed voters to have irrational preferences. Given three candidates, $c$, $d$, and $e$, it was perfectly acceptable for a voter to prefer $c$ to $d$, $d$ to $e$, and $e$ to $c$. On the other hand, in modern studies of voting and election systems each voter’s preferences are most typically modeled as a linear order over all candidates (In this paper, as is standard, “linear order” implies strictness, i.e., no tie in the ordering.) Yet irrationality is a very tempting and natural concept. Consider Bob, who likes to eat out but is often in a hurry. Bob prefers diners to fast food because he is willing to wait a little more to get better food. Also, given a choice between a fancy restaurant and a diner he prefers the restaurant, again because he is willing to wait somewhat longer to get better quality. However, given the choice between a fast-food place and a fancy restaurant Bob might reason that he is not willing to wait so much longer to be served at the fancy restaurant and so will choose fast food instead. Thus regarding catering options, Bob’s preferences are irrational in our sense. Similar irrationalities easily come up when voters
make their choices based on multiple criteria—a very natural scenario both among humans and software agents.

Llull’s election system is remarkably similar to what is now known as “Copeland elections” [Cop51], a more than half-century old voting procedure that is based on pairwise comparisons of candidates: The winner (by a majority of votes—in this paper “majority” always, as is standard, means strict majority) of each such a head-to-head contest is awarded one point and the loser receives no point; in ties, both parties are (in the most common interpretation of Copeland’s meaning) awarded half a point; whoever collects the most points over all these contests (including tie-related points) is the election’s winner. In fact, the points awarded for ties in such head-to-head majority-rule contests are treated in two ways, half a point (most common) and zero points (uncommon), in the literature when speaking of Copeland elections. To provide a framework that can capture both those notions, as well as Llull’s system and the whole family of systems created by choices of how we valve ties, we propose and introduce a parameterized version of Copeland elections, denoted by Copeland\(\alpha\), where the parameter \(\alpha\) is a rational number, \(0 \leq \alpha \leq 1\), and in the case of a tie both candidates receive \(\alpha\) points. So the system widely referred to in the literature as “Copeland elections” is Copeland\(0.5\), where tied candidates receive half a point each (see, e.g., Merlin and Saari [SM96,MS97]; the definition used by Conitzer et al. [CSL07] can be scaled to be equivalent to Copeland\(0.5\)). Copeland\(0\), where tied candidates come away empty-handed, has sometimes also been referred to as “Copeland elections” (see, e.g., Procaccia, Rosenschein, and Kaminka [PRK07] and an early version of this paper [FHHR07]). The above-mentioned election system proposed by Ramon Llull in the thirteenth century is in this notation nothing other than Copeland\(1\), where tied candidates are awarded one point each, just like winners of head-to-head contests.\(^1\) The group stage of the FIFA World Cup finals is in essence a collection of Copeland\(\alpha\) tournaments with \(\alpha = \frac{1}{3}\).

At first glance, one might be tempted to think that the definitional perturbation due to the parameter \(\alpha\) in Copeland\(\alpha\) elections is negligible. However, it in fact can make the dynamics of Llull’s system quite different from those of, for instance, Copeland\(0.5\) or Copeland\(0\). Specific examples witnessing this claim, both regarding complexity results and regarding their proofs, are given at the end of Section 1.3.

Finally, we mention that a probabilistic variant of Copeland voting (known as the Jech method) was defined already in 1929 by Zermelo [Zer29] and later on was reintroduced by several other researches (see, e.g., the paper of Levin and Nalebuff [LN95] for further references and a description of the Jech method). We note in passing that the Jech method is applicable even when we feed it incomplete information. In the present paper, however,

\(^1\)Page 23 of Hägele and Pukelsheim [HP01] indicates in a way we find deeply convincing (namely by a direct quote of Llull’s in-this-case-very-clear words from his Artifitium Electionis Personarum—which was rediscovered by those authors in the year 2000) that at least one of Llull’s election systems was Copeland\(1\), and so in this paper we refer to the both-candidates-score-a-point-on-a-tie variant as Llull voting.

In some settings Llull required the candidate and voter sets to be identical and had an elaborate two-stage tie-breaking rule ending in randomization. We disregard these issues here and cast his system into the modern idiom for election systems.
we do not consider incomplete-information or probabilistic scenarios, although we commend such settings as interesting for future work.

1.2 Computational Social Choice

In general it is impossible to design a perfect election system. In the 1950s Arrow [Arr63] famously showed that there is no social choice system that satisfies a certain small set of reasonable requirements, and later Gibbard [Gib73], Satterthwaite [Sat75], and Duggan and Schwartz [DS00] showed that any natural election system can be manipulated by strategic voting, i.e., by a voter who reveals different preferences than his or her true ones in order to affect an election's result in his or her favor. Also, no natural election system with a polynomial-time winner-determination procedure has yet been shown to be resistant to all types of control via procedural changes. Control refers to attempts by an external agent (called “the chair”) to, via such actions as addition/deletion/partition of candidates or voters, make a given candidate win the election (in the case of constructive control [BTT92]) or preclude a given candidate’s victory (in the case of destructive control [HHR07a]).

These obstacles are very discouraging, but the field of computational social choice theory grew in part from the realization that computational complexity provides a tool to partially circumvent these obstacles. In particular, around 1990 Bartholdi, Tovey, and Trick [BTT89a,BTT92] and Bartholdi and Orlin [BO91] brilliantly observed that while we might not be able to make manipulation (i.e., strategic voting) and control of elections impossible, we can at least try to make such manipulation and control so computationally difficult that neither voters nor election organizers will attempt it. For example, if there is a way for a committee’s chair to set up an election within the committee in such a way that his or her favorite option is guaranteed to win but the chair’s computational task would take a million years, then for all practical purposes we may assume that the chair is prevented from finding such a set-up.

Since the seminal work of Bartholdi, Orlin, Tovey, and Trick a large body of research has been dedicated to the study of computational properties of election systems. Some topics that have received much attention are the complexity of manipulating elections [CS03,CS06,CSL07,EL05,HH07,PR07,PRZ07] and of controlling elections via procedural changes [HHR07a,HHR07b,PRZ07]. Recently, Faliszewski, Hemaspaanda, and Hemaspaandra introduced the study of the complexity of bribery in elections ([FHH06a], see also [Fal08]). Bribery shares some features of manipulation and some features of control. In particular, the briber picks the voters he or she wants to affect (as in voter control problems) and asks them to vote as he or she wishes (as in manipulation). (For additional citations and pointers, see the recent survey [FHHR].)

In this paper we study Copeland$^\alpha$ elections with respect to the computational complexity of bribery and procedural control.$^2$ The study of election systems and their computational properties, such as the complexity of their manipulation, control, and bribery problems, is an important topic in multiagent systems. Agents (as voters are called in this context)

$^2$See [FHS08] for a study of manipulation within Copeland$^\alpha$. 

4
may have different, often conflicting, individual preferences over the given alternatives (or candidates) and voting rules (or, synonymously, election systems) provide a useful method for them to come to a “reasonable” decision on which alternative to choose. Thus elections can be employed in multiagent settings and also in other contexts to solve many practical problems. As just a few examples we mention the work of Ephrati and Rosenschein [ER97] where elections are used for planning, the work of Ghosh et al. [GMHS99] who developed a recommender system for movies that is based on voting, and the work of Dwork et al. [DKNS01] where elections are used to aggregate results from multiple web-search engines. In a multiagent setting we may have hundreds of elections happening every minute and we cannot hope to carefully check in each case whether the party that organized the election attempted some procedural change to skew the results. However, if it is computationally hard to effectuate such procedural changes then we can hope it is practically infeasible for the organizers to undertake them.

A standard technique for showing that a particular election-related problem (e.g., the problem of deciding whether the chair can make his or her favorite candidate a winner by influencing at most $k$ voters not to cast their votes) is computationally intractable is to show that it is NP-hard. This approach is taken in almost all of the papers on computational social choice cited above, and it is the approach that we take in this paper. One of the justifications for using NP-hardness as a barrier against manipulation and control of elections is that in multiagent settings any attempts to influence the election’s outcome are made by computationally bounded software agents that have neither human intuition nor the computational ability to solve NP-hard problems.

Recently, such papers as [MPS08, PR07, CS06, HH] have studied the frequency (or sometimes, probability weight) of correctness of heuristics for voting problems. We view worst-case study as a natural prerequisite to a frequency-of-hardness attack: After all, there is no point in seeking frequency-of-hardness results if the problem at hand is in P to begin with. And if one cannot even prove worst-case hardness for a problem, then proving average-case hardness is even more beyond reach. Also, current frequency results have debilitating limitations (for example, being locked into specific distributions; depending on unproven assumptions; adopting “tractability” notions that declare undecidable problems tractable and that are not robust under even linear-time reductions). Although frequency of hardness is a fascinating and important direction, these models are arguably not ready for prime time and, contrary to some people’s impression, fail to imply average-case polynomial runtime claims. [EHRS07, HH] provide discussion of some of these issues.

1.3 Outline of Our Results

The goal of this paper is to study Copeland$^\alpha$ elections from the point of view of computational social choice theory, in the setting where voters are rational and in the setting where the voters are allowed to have irrational preferences. (Note: When we henceforth say “irrational voters,” we mean that the voters may have irrational preferences, not that they each must.) We study the issues of bribery and control and we point the reader to the work of Faliszewski, Hemaspaandra, and Schnoor [FHS08] for work on manipulation.
One of the major achievements of this paper is to classify which of resistance or vulnerability holds for Copeland$^\alpha$ in every previously studied control scenario for each rational value of $\alpha$. In doing so, we provide an example of a control problem where the complexity of Copeland$^{0.5}$ (which is the system commonly referred to as “Copeland”) differs from that of both Copeland$^0$ and Copeland$^1$: While the latter two problems are vulnerable to constructive control by adding (an unlimited number of) candidates, Copeland$^{0.5}$ is resistant to this control type (see Section 2 for definitions and Theorem 4.10 for this result).

Thus Copeland (i.e., Copeland$^{0.5}$) is the first natural election system with a polynomial-time winner problem that is proven to be resistant to every type of constructive control that has been proposed in the literature to date. This result closes a 15-year-old quest for a natural election system fully resistant to constructive control.

We also show that Copeland$^\alpha$ is resistant to both constructive and destructive bribery, both for the case of rational and irrational voters. Our hardness proofs work for the case of unweighted voters without price tags (see [FHH06a]) and thus, naturally, apply as well to the more involved scenarios of weighted unpriced voters, unweighted priced voters, and weighted priced voters.

To prove our bribery results, we introduce a method of controlling the relative performances of certain voters in such a way that, if one sets up other restrictions appropriately, the legal possibilities for bribery actions are sharply constrained. We call our approach “the UV technique,” since it is based on dummy candidates $u$ and $v$. The proofs of Theorems 3.2 and 3.4 are particular applications of this method. We feel that the UV technique will be useful, even beyond the scope of this paper, for the analysis of bribery in other election systems based on head-to-head contests.

We also study Copeland$^\alpha$ elections under more flexible models such as “microbribery” (see Section 3.2) and “extended control” (see Section 4.3). We show that Copeland$^\alpha$ (with irrational voters allowed) is vulnerable to destructive microbribery and to destructive candidate control via providing fairly simple greedy algorithms. In contrast, our polynomial-time algorithms for constructive microbribery are proven via a technique involving min-cost network flows. To the best of our knowledge, this is one of the first applications of this technique to election problems.\(^3\) We believe that the range of applicability of flow networks to election problems extends well beyond microbribery for Copeland$^\alpha$ elections and we point the reader to a recent paper by Procaccia, Rosenschein, and Zohar [PRZ08] and to a paper by Faliszewski [Fal08] for examples of such applications.

We also mention that during our study of Copeland control we have noticed that the proof of an important result of Bartholdi, Tovey, and Trick [BTT92, Theorem 12] (namely, that Condorcet voting is resistant to constructive control by deleting voters) is invalid. The invalidity is due to the proof centrally using nonstrict voters, in violation of Bartholdi, Tovey, and Trick’s [BTT92] (and our) model, and the invalidity seems potentially daunting to seek to fix with the proof approach taken there. We noticed also that Theorem 14 of the same paper has a similar flaw. In Section 5 we validly reprove their claimed results using

\(^3\)Curiously, we recently learned that Procaccia, Rosenschein, and Zohar [PRZ08] independently used a similar technique in their work regarding the complexity of achieving proportional representation.
As mentioned in Section 1.1, Copeland\textsuperscript{α} elections may behave quite differently depending on the value of the tie-rewarding parameter \( \alpha \), and we now give concrete examples to make this case. Specifically, proofs of results for Copeland\textsuperscript{α} occasionally differ considerably for distinct values of \( \alpha \), and in some cases even the computational complexity of various control and manipulation problems (for the latter see [FHS08]) may jump between P membership and NP-completeness depending on \( \alpha \). Regarding control, we have already noted that Theorem 4.10 shows that some control problem (namely, control by adding an unlimited number of candidates) for Copeland\textsuperscript{α} is NP-complete for each rational \( \alpha \) with \( 0 < \alpha < 1 \), yet Theorem 4.11 shows that same control problem to be in P for \( \alpha \in \{0,1\} \). To give another example involving a different control problem (namely control by partition of candidates with the ties-eliminate tie-handling rule, see Section 2), note that the proofs of Theorem 4.21 (which applies to \( \alpha = 1 \) for this control problem within Copeland\textsuperscript{α}) and of Theorem 4.22 (which applies to all rational \( \alpha \) with \( 0 \leq \alpha < 1 \) for the same problem) differ substantially. Regarding constructive microbribery, the vulnerability constructions for \( \alpha = 0 \) (see Lemma 3.13) and \( \alpha = 1 \) (see Lemma 3.16) significantly differ from each other and neither of them works for tie-rewarding values other than 0 and 1. The above remarks notwithstanding, for most of our results we show that it is possible to obtain a unified—though due to this uniformity sometimes rather involved—construction that works for Copeland\textsuperscript{α} with respect to every rational \( \alpha, 0 \leq \alpha \leq 1 \).

1.4 Organization

This paper is organized as follows. In Section 2, we formalize the notion of elections and, in particular, of Copeland\textsuperscript{α} elections, we introduce some useful notation, and we formally define the control and bribery problems we are interested in. In Section 3, we show that for each rational \( \alpha, 0 \leq \alpha \leq 1 \), Copeland\textsuperscript{α} elections are fully resistant to bribery, both in the case of rational voters and in the case of irrational voters. On the other hand, if one changes the bribery model to allow microbribes of voters, we prove vulnerability for each rational \( \alpha, 0 \leq \alpha \leq 1 \), in the irrational-voters destructive case and for some specific values of \( \alpha \) in the irrational-voters constructive case. In Sections 4.1 and 4.2, we present our results on procedural control for Copeland\textsuperscript{α} elections for each rational \( \alpha \) with \( 0 \leq \alpha \leq 1 \). Section 4.3 presents our results on fixed-parameter tractability of bounded-case control for Copeland\textsuperscript{α}. Section 5 provides valid proofs for several control problems for Condorcet elections studied by Bartholdi, Tovey, and Trick [BTT92]. We conclude the paper with a brief summary in Section 6 and by stating some open problems.

2 Preliminaries

2.1 Elections: The Systems of Llull and Copeland

An election \( E = (C, V) \) consists of a finite candidate set \( C = \{c_1, \ldots, c_n\} \) and a finite collection \( V \) of voters, where each voter is represented (individually, except later when we
discuss succinct inputs) via his or her preferences over the candidates. An election system is a rule that determines the winner(s) of each given election.

We consider two ways in which voters can express their preferences. In the rational case, each voter’s preferences are represented as a linear order over the set $C$, i.e., each voter $v_i$ has a preference list $c_{i_1} > c_{i_2} > \cdots > c_{i_n}$, with $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$. In the irrational case, each voter’s preferences are represented as a table that for every unordered pair of distinct candidates $c_i$ and $c_j$ in $C$ indicates whether the voter prefers $c_i$ to $c_j$ (i.e., $c_i > c_j$) or $c_j$ to $c_i$ (i.e., $c_j > c_i$).

Some well-known election rules for the case of rational voters include plurality, Borda count, and Condorcet. Plurality elects the candidate(s) that are ranked first by the largest number of voters. Borda count elects the candidate(s) that receive the most points, where each voter $v_i$ gives each candidate $c_j$ as many points as the number of candidates $c_j$ is preferred to with respect to $v_i$’s preferences. A Condorcet winner is a candidate $c_i$ such that for every other candidate $c_j$ it holds that $c_i$ is preferred to $c_j$ by a majority of voters.

In this paper, we introduce a parameterized version of Copeland’s election system [Cop51]. We denote our parametrized version (family) by Copeland $\alpha$, where the parameter $\alpha$ is a rational number between 0 and 1 that specifies how ties are rewarded in the head-to-head majority-rule contests between any two distinct candidates.

**Definition 2.1** Let $\alpha$, $0 \leq \alpha \leq 1$, be a fixed rational number. In a Copeland $\alpha$ election, the voters indicate which among each pair of distinct candidates they prefer. For each such head-to-head contest, if some candidate is preferred by a majority of voters then he or she obtains one point and the other candidate obtains zero points, and if a tie occurs then both candidates obtain $\alpha$ points. Let $E = (C, V)$ be an election. For each $c \in C$, $\text{score}_E^\alpha(c)$ is the sum of $c$’s Copeland $\alpha$ points in $E$. Every candidate $c$ with maximum $\text{score}_E^\alpha(c)$ (i.e., $(\forall d \in C)[\text{score}_E^\alpha(c) \geq \text{Copeland}_E^\alpha(d)]$) wins.

Let Copeland $\alpha_{\text{Irrational}}$ denote the same election system but with voters allowed to be irrational.

In the literature, the term “Copeland elections” is most often used for the system Copeland$^0$, but has occasionally been used for Copeland$^0$. As mentioned earlier, the system Copeland$^1$ was proposed by Llull already in the thirteenth century (see the literature pointers given in the introduction) and so is called Llull voting.

We now define some notation to facilitate the discussion of Copeland$^\alpha$ elections. Let $E = (C, V)$ be an election and let $c_i$ and $c_j$ be two arbitrary candidates from $C$. By $\text{vs}_E(c_i, c_j)$ we mean the surplus of votes that candidate $c_i$ has over $c_j$. Formally, we define $\text{vs}_E(c_i, c_j)$ to be 0 if $c_i = c_j$, and otherwise to be

$$\text{vs}_E(c_i, c_j) = \|\{v \in V \mid v \text{ prefers } c_i \text{ to } c_j\}\| - \|\{v \in V \mid v \text{ prefers } c_j \text{ to } c_i\}\|.$$
That is, if \( c_i \) defeats \( c_j \) in a head-to-head contest in \( E \) then \( \text{vs}_E(c_i, c_j) > 0 \), if they are tied then \( \text{vs}_E(c_i, c_j) = 0 \), and if \( c_j \) defeats \( c_i \) then \( \text{vs}_E(c_i, c_j) < 0 \). (Throughout this paper, “defeats” excludes the possibility of a tie, i.e., “defeats” means “(strictly) defeats.” We will say ties-or-defeats when we wish to allow a tie to suffice.) Clearly, \( \text{vs}_E(c_i, c_j) = - \text{vs}_E(c_j, c_i) \).

We call the value \( \text{vs}_E(c_i, c_j) \) a relative vote-score of \( c_i \) with respect to \( c_j \). We often speak, in the plural, of relative vote-scores when we mean a group of results of head-to-head contests between particular candidates.

Let \( \alpha, 0 \leq \alpha \leq 1 \), be a fixed rational number. Definition 2.1 introduced \( \text{score}_E^\alpha(c) \), the Copeland\(^\alpha \) score of candidate \( c \) within election \( E \). Note that for each candidate \( c_i \in C \),

\[
\text{score}_E^\alpha(c_i) = \| \{ c_j \in C \mid c_i \neq c_j \text{ and } \text{vs}_E(c_i, c_j) > 0 \} \| + \alpha \| \{ c_j \in C \mid c_i \neq c_j \text{ and } \text{vs}_E(c_i, c_j) = 0 \} \|.
\]

In particular, we have \( \text{score}_E^0(c_i) = \| \{ c_j \in C \mid c_i \neq c_j \text{ and } \text{vs}_E(c_i, c_j) > 0 \} \| \), and \( \text{score}_E^1(c_i) = \| \{ c_j \in C \mid c_i \neq c_j \text{ and } \text{vs}_E(c_i, c_j) \geq 0 \} \| \). Note further that the highest potential Copeland\(^\alpha \) score in any election \( E = (C, V) \) is \( \| C \| - 1 \).

A candidate \( c_i \in C \) is a Copeland\(^\alpha \) winner of \( E = (C, V) \) if for all \( c_j \in C \) it holds that \( \text{score}_E^\alpha(c_i) \geq \text{score}_E^\alpha(c_j) \). (Note that more than one candidate may tie as Copeland\(^\alpha \) winners.) A candidate \( c_i \) is a Condorcet winner of \( E \) if \( \text{score}_E^0(c_i) = \| C \| - 1 \), that is, exactly if \( c_i \) defeats all other candidates in head-to-head contests.

In many of our constructions to be presented in the upcoming proofs, we use the following notation for rational voters.

**Notation 2.2** Within every election we fix some arbitrary order over the candidates. Any occurrence of a subset \( D \) of candidates in a preference list means the candidates from \( D \) are listed with respect to that fixed order. Occurrences of \( \overline{D} \) mean the same except that the candidates from \( D \) are listed in the reverse order.

For example, if \( C = \{ a, b, c, d, e \} \), with the order being first “a” then “b” . . . then “e”, and \( D = \{ a, c, e \} \) then \( b > D > d \) means \( b > a > c > e > d \) and \( b > \overline{D} > d \) means \( b > e > c > a > d \).

### 2.2 Bribery and Control Problems

We now describe the computational problems that we study in this paper. Our problems come in two flavors: constructive and destructive. In the constructive version the goal is to test whether, via either bribery or control, it is possible to make a given candidate a winner of the election. In the destructive case the goal is to test whether it is possible to prevent a given candidate from being a winner of the election.

Let \( E \) be an election system. In our case, \( E \) will be either Copeland\(^\alpha \) or Copeland\(^\alpha\)\(_{\text{Irrational}} \) where \( \alpha, 0 \leq \alpha \leq 1 \), is a fixed rational number. The bribery problem for \( E \) with rational voters is defined as follows [FHH06a].

**Name:** \( E \)-bribery and \( E \)-destructive-bribery.
Given: A set $C$ of candidates, a collection $V$ of voters specified via their preference lists over $C$, a distinguished candidate $p \in C$, and a nonnegative integer $k$.

Question (constructive): Is it possible to make $p$ a winner of the $E$ election $(C, V)$ by modifying the preference lists of at most $k$ voters?

Question (destructive): Is there some modification of the preference lists of at most $k$ voters under which $p$ is not a winner of the $E$ election.

The version of this problem for elections with irrational voters allowed is defined exactly like the rational one, with the only difference being that voters are represented via preference tables rather than preference lists (at the end of the present section, Section 2.2, we make the analogous comment for the unique-winner cases), and the briber may completely change a voter’s preference table at unit cost. Later in the paper we will study another variant of bribery problems—a variant in which one is allowed to perform microbribes: bribes the costs of which depend on each preference-table entry change and the briber pays separately for each such change.

Bribery problems seek to change the outcome of elections via modifying the reported preferences of some of the voters. Control problems, however, seek to change the outcome by modifying the elections’ structure via adding/deleting/partitioning either candidates or voters. When formally defining these control types, we use the following naming conventions for the corresponding control problems. The name of a control problem starts with the election system used (when clear from context, it may be dropped), followed by CC for “constructive control” or by DC for “destructive control,” followed by the acronym of the type of control: AC for “adding (a limited number of) candidates,” AC$_u$ for “adding (an unlimited number of) candidates,” DC for “deleting candidates,” PC for “partition of candidates,” RPC for “run-off partition of candidates,” AV for “adding voters,” DV for “deleting voters,” and PV for “partition of voters.” As defined by Bartholdi, Tovey, and Trick [BTT92], all the partitioning cases (PC, RPC, and PV) are two-stage elections, and we here adopt the tie-handling rules proposed by Hemaspaandra, Hemaspaandra, and Rothe [HHR07a] for first-stage subelections in these two-stage elections. So, for all the partitioning cases, the acronym PC, RPC, and PV, respectively, is followed by the acronym of the tie-handling rule used in first-stage subelections, namely TP for “ties promote” (i.e., all winners of a given first-stage subelection are promoted to the final round of the election) and TE for “ties eliminate” (i.e., only the unique winner of a given first-stage subelection is promoted to the final round of the election, and if there is more than one winner in a given first-stage subelection then none of this subelection’s winners are promoted to the final round).

We now formally define our control problems. These definitions are due to Bartholdi, Tovey, and Trick [BTT92] for constructive control and to Hemaspaandra, Hemaspaandra, and Rothe [HHR07a] for destructive control.

Let $E$ be an election system. Again, $E$ will here be either Copeland$^\alpha$ or Copeland$^{\alpha\text{irrational}}$, where $\alpha$, $0 \leq \alpha \leq 1$, is a fixed rational number. When we now define our control problems, we focus on the case of rational voters.
Control via Adding Candidates

We start with two versions of control via adding candidates. In the unlimited version the goal of the chair of the election is to introduce candidates from a pool of spoiler candidates so as to make his or her favorite candidate a winner of the election (in the constructive case) or prevent his or her despised candidate from being a winner (in the destructive case). As suggested by the name of the problem, in the unlimited version the chair can introduce any subset (none, some, or all are all legal options) of the spoiler candidates into the election.

Name: $\mathcal{E}$-CCAC$_u$ and $\mathcal{E}$-DCAC$_u$ (control via adding an unlimited number of candidates).

Given: Disjoint sets $C$ and $D$ of candidates, a collection $V$ of voters specified via their preference lists over the candidates in the set $C \cup D$, and a distinguished candidate $p \in C$.

Question ($\mathcal{E}$-CCAC$_u$): Is it possible to choose a subset $E$ of $D$ such that $p$ is a winner of the $\mathcal{E}$ election with voters $V$ and candidates $C \cup E$?

Question ($\mathcal{E}$-DCAC$_u$): Is it possible to choose a subset $E$ of $D$ such that $p$ is not a winner of the $\mathcal{E}$ election with voters $V$ and candidates $C \cup E$?

The above definition of $\mathcal{E}$-CCAC$_u$ is based on that introduced by Bartholdi, Tovey, and Trick [BTT92]. In contrast with the other control problems involving adding or deleting candidates or voters, in the adding candidates problem Bartholdi, Tovey, and Trick did not introduce a nonnegative integer $k$ that bounds the number of candidates (from the set $D$) the chair is allowed to add. We feel this asymmetry in their definitions is not well-justified and thus we define a with-change-parameter version of the control-by-adding-candidates problems, which we denote by AC$_l$ (where the “$l$” stands for the fact that part of the problem is a limit on the number of candidates that can be added, in contrast with the model of Bartholdi, Tovey, and Trick [BTT92], which we denote AC$_u$ with the “$u$” standing for the fact that the number of added candidates is unlimited, at least in the sense of not being limited via a separately input integer). The with-parameter version is the long-studied case for AV, DV, and DC, and we in this paper will use AC as being synonymous with AC$_l$, and will thus use the notation AC for the rest of this paper when speaking of AC. We suggest this as a natural regularization of the definitions and we hope this version will become the “normal” version of the problem for further study. However, we caution the reader that in earlier papers AC is used to mean AC$_u$. Indeed, in the present paper, we will obtain results not just for AC$_l$ but also for the AC$_u$ case, in order to allow comparisons between the results of this paper and those of earlier works. So, turning now to what we mean by AC (equivalently, AC$_l$), as per the above definition in $\mathcal{E}$-CCAC (equivalently, $\mathcal{E}$-CCAC$_l$) we ask whether it is possible to make the distinguished candidate $p$ a winner of some $\mathcal{E}$ election obtained by adding at most $k$ candidates from the spoiler candidate set $D$. (Note that $k$ is part of the input.) We define the destructive version, $\mathcal{E}$-DCAC (equivalently, $\mathcal{E}$-DCAC$_l$), analogously.

Name: $\mathcal{E}$-CCAC and $\mathcal{E}$-DCAC (control via adding a limited number of candidates).
Given: Disjoint sets $C$ and $D$ of candidates, a collection $V$ of voters specified via their preference lists over the candidates in the set $C \cup D$, a distinguished candidate $p \in C$, and a nonnegative integer $k$.

**Question (E-CCAC):** Is it possible to choose a subset $E$ of $D$ such that $|E| \leq k$ and $p$ is a winner of the $E$ election with voters $V$ and candidates $C \cup E$?

**Question (E-DCAC):** Is it possible to choose a subset $E$ of $D$ such that $|E| \leq k$ and $p$ is not a winner of the $E$ election with voters $V$ and candidates $C \cup E$?

**Control via Deleting Candidates**

In (constructive) control via deleting candidates, the chair seeks to ensure that his or her favorite candidate $p$ is a winner of the election by suppressing at most $k$ candidates. In the destructive variant of this problem, the chair’s goal is to block $p$ from winning by suppressing at most $k$ candidates other than $p$.

**Name:** $E$-CCDC and $E$-DCDC (control via deleting candidates).

**Given:** A set $C$ of candidates, a collection $V$ of voters represented via preference lists over $C$, a distinguished candidate $p \in C$, and a nonnegative integer $k$.

**Question (E-CCDC):** Is it possible to by deleting at most $k$ candidates ensure that $p$ is a winner of the resulting $E$ election?

**Question (E-DCDC):** Is it possible to by deleting at most $k$ candidates other than $p$ ensure that $p$ is not a winner of the resulting $E$ election?

**Control via Partition and Run-Off Partition of Candidates**

Bartholdi, Tovey, and Trick [BTT92] gave two types of control of elections via partition of candidates. In both cases the candidate set $C$ is partitioned into two groups, $C_1$ and $C_2$, and the election is conducted in two stages. For control via run-off partition of candidates, the election’s first stage is conducted separately within each group of candidates, $C_1$ and $C_2$, and the group winners that survive the tie-handling rule compete against each other in the second stage. In control via partition of candidates, the first-stage election is performed only within $C_1$ and those of that election’s winners that survive the tie-handling rule compete against all candidates from $C_2$ in the second stage.

We follow the two tie-handling rules proposed by Hemaspaandra, Hemaspaandra, and Rothe [HHR07a]. In the ties-promote (TP) model, all the first-stage winners within a group are promoted to the final round. In the ties-eliminate (TE) model, a first-stage winner within a group is promoted to the final round if and only if he or she is the unique winner within that group.

**Name:** $E$-CCRPC and $E$-DCRPC (control via run-off partition of candidates).
Given: A set $C$ of candidates, a collection $V$ of voters represented via preference lists over $C$, and a distinguished candidate $p \in C$.

Question (E-CCRPC): Is it possible to partition $C$ into $C_1$ and $C_2$ such that $p$ is a winner of the two-stage election where the winners of subelection $(C_1, V)$ that survive the tie-handling rule compete against the winners of subelection $(C_2, V)$ that survive the tie-handling rule? Each subelection (in both stages) is conducted using election system $E$.

Question (E-DCRPC): Is it possible to partition $C$ into $C_1$ and $C_2$ such that $p$ is not a winner of the two-stage election where the winners of subelection $(C_1, V)$ that survive the tie-handling rule compete against the winners of subelection $(C_2, V)$ that survive the tie-handling rule? Each subelection (in both stages) is conducted using election system $E$.

The above description defines four computational problems for a given election system $E$: $E$-CCRPC-TE, $E$-CCRPC-TP, $E$-DCRPC-TE, and $E$-DCRPC-TP.

Name: $E$-CCPC and $E$-DCPC (control via partition of candidates).

Given: A set $C$ of candidates, a collection $V$ of voters represented via preference lists over $C$, and a distinguished candidate $p \in C$.

Question (E-CCPC): Is it possible to partition $C$ into $C_1$ and $C_2$ such that $p$ is a winner of the two-stage election where the winners of subelection $(C_1, V)$ that survive the tie-handling rule compete against all candidates in $C_2$? Each subelection (in both stages) is conducted using election system $E$.

Question (E-DCPC): Is it possible to partition $C$ into $C_1$ and $C_2$ such that $p$ is not a winner of the two-stage election where the winners of subelection $(C_1, V)$ that survive the tie-handling rule compete against all candidates in $C_2$? Each subelection (in both stages) is conducted using election system $E$.

As above, this description defines four computational problems for a given election system $E$: $E$-CCPC-TE, $E$-CCPC-TP, $E$-DCPC-TE, and $E$-DCPC-TP.

Control via Adding Voters

In the scenario of control via adding voters, the chair’s goal is to either ensure that $p$ is a winner (in the constructive case) or ensure that $p$ is not a winner (in the destructive case) via causing up to $k$ additional voters to participate in the election. The chair can draw the voters to add to the election from a prespecified collection of voters (with given preferences).

Name: $E$-CCAV and $E$-DCAV (control via adding voters).
**Given:** A set $C$ of candidates, two disjoint collections of voters, $V$ and $W$, represented via preference lists over $C$, a distinguished candidate $p$, and a nonnegative integer $k$.

**Question ($\mathcal{E}$-CCA V):** Is it possible to select a subset $Q$, $\|Q\| \leq k$, of voters in $W$ such that the voters in $V \cup Q$ jointly elect $p \in C$ as a winner according to the system $\mathcal{E}$?

**Question ($\mathcal{E}$-DCA V):** Is it possible to select a subset $Q$, $\|Q\| \leq k$, of voters in $W$ such that the voters in $V \cup Q$ do not elect $p$ as a winner according to the system $\mathcal{E}$?

**Control via Deleting Voters**

In the control via deleting voters case the chair seeks to either ensure that $p$ is a winner (in the constructive case) or prevent $p$ from being a winner (in the destructive case) via blocking up to $k$ voters from participating in the election. (This loosely models vote suppression or disenfranchisement.)

**Name:** $\mathcal{E}$-CCDV and $\mathcal{E}$-DCDV (control via deleting voters).

**Given:** A set $C$ of candidates, a collection $V$ of voters represented via preference lists over $C$, a distinguished candidate $p \in C$, and a nonnegative integer $k$.

**Question ($\mathcal{E}$-CCDV):** Is it possible to by deleting at most $k$ voters ensure that $p$ is a winner of the resulting $\mathcal{E}$ election?

**Question ($\mathcal{E}$-DCDV):** Is it possible to by deleting at most $k$ voters ensure that $p$ is not a winner of the resulting $\mathcal{E}$ election?

**Control via Partition of Voters**

In the case of control via partition of voters, the following two-stage election is performed. First, the voter set $V$ is partitioned into two subcommittees, $V_1$ and $V_2$. The winners of election $(C, V_1)$ that survive the tie-handling rule compete against the winners of $(C, V_2)$ that survive the tie-handling rule. Again, our tie-handling rules are either TE or TP (ties-eliminate or ties-promote).

**Name:** $\mathcal{E}$-CCPV and $\mathcal{E}$-DCPV (control via partition of voters).

**Given:** A set $C$ of candidates, a collection $V$ of voters represented via preference lists over $C$, and a distinguished candidate $p \in C$.

**Question ($\mathcal{E}$-CCPV):** Is it possible to partition $V$ into $V_1$ and $V_2$ such that $p$ is a winner of the two-stage election where the winners of election $(C, V_1)$ that survive the tie-handling rule compete against the winners of $(C, V_2)$ that survive the tie-handling rule? Each subelection (in both stages) is conducted using election system $\mathcal{E}$.
**Question (E-DCPV):** Is it possible to partition $V$ into $V_1$ and $V_2$ such that $p$ is not a winner of the two-stage election where the winners of election $(C, V_1)$ that survive the tie-handling rule compete against the winners of $(C, V_2)$ that survive the tie-handling rule? Each subelection (in both stages) is conducted using election system $E$.

**Unique Winners and Irrationality**

Our bribery and control problems were each defined above for rational voters only and in the nonunique-winner model, i.e., asking whether a give candidate can be made, or prevented from being, a winner. Nonetheless, we have proven all our control results both for the case of nonunique winners and (to be able to fairly compare them with existing control results, which except for the interesting “multi-winner” model of Procaccia, Rosenschein, and Zohar [PRZ07] are in the unique-winner model) unique winners, i.e., asking whether a given candidate can be made, or prevented from being, a sole winner. However, to stay in sync with the existing literature on bribery, we prove the bribery results only in the nonunique-winner model. In addition to the rational-voters case, we also study these problems for the case of voters who are allowed to be irrational. In the case of irrational voters we assume that all voters are represented via preference tables rather than preference lists.

**2.3 Graphs**

An undirected graph $G$ is a pair $(V, E)$, where $V$ is the set of vertices and $E$ is the set of edges and each edge is an unordered pair of two distinct vertices. A directed graph is defined analogously, except that the edges are represented as ordered pairs. For example, if $u$ and $v$ are two distinct vertices in an undirected graph $G$ then $G$ either has an edge $e = \{u, v\}$ that connects $u$ and $v$ or it doesn’t. On the other hand, if $G$ is a directed graph then $G$ either has an edge $e' = (u, v)$ from $u$ to $v$, or an edge $e'' = (v, u)$ from $v$ to $u$, or both $e'$ and $e''$, or neither $e'$ nor $e''$.

The indegree of a vertex $u$ in a given graph, denoted by $\text{deg}_{\text{in}}(u)$, is the number of edges that enter $u$. Similarly, the outdegree of $u$ in a given graph, denoted by $\text{deg}_{\text{out}}(u)$, is the number of edges that leave $u$.

Except two paragraphs ago, the symbols $E$ and $V$ are generally reserved in this paper to represent, respectively, elections and collections of voters. However, we will “overload” these symbols in the following way. If $G$ is a graph (directed or undirected) then $V(G)$ is the set of vertices of $G$ and $E(G)$ is the set of edges of $G$. Many of our proofs involve several different graphs and elections at the same time, and this convention will help us avoid confusion and will save us the effort of making up new names.

**2.4 NP-Complete Problems and Reductions**

Without loss of generality, we assume that all problems that we consider are encoded in a natural, efficient way over the alphabet $\Sigma = \{0, 1\}$. We use the standard notion
of NP-completeness, defined via polynomial-time many-one reductions. We say that a computational problem A \textit{polynomial-time many-one reduces} to a problem B if there exists a polynomial-time computable function f such that

\[(\forall x \in \Sigma^*)(x \in A \iff f(x) \in B).\]

A problem is \textit{NP-hard} if all members of NP polynomial-time many-one reduce to it. A problem is \textit{NP-complete} if it is NP-hard and is a member of NP. When clear from context we will use “reduce” and “reduction” as shorthands for “polynomial-time many-one reduce” and “polynomial-time many-one reduction.”

Our NP-completeness results typically follow via a reduction from either the exact-cover-by-3-sets problem or from the vertex cover problem (see, e.g., [GJ79]). These are well-known NP-complete problems, but we define them here for the sake of completeness.

**Name:** X3C (exact cover by 3-sets).

**Given:** A set \(B = \{b_1, \ldots, b_{3k}\}\) and a family of sets \(S = \{S_1, \ldots, S_n\}\) such that for each \(i, 1 \leq i \leq n\), it holds that \(S_i \subseteq B\) and \(\|S_i\| = 3\).

**Question:** Is there a size-\(k\) set \(A \subseteq \{1, \ldots, n\}\) such that \(\bigcup_{i \in A} S_i = B\)?

Whenever we consider instances of the X3C problem, we assume that they are well-formed, that is, we assume that they follow the syntactic requirements stated in the above “Given” field (e.g., the cardinality of the set \(B\) is indeed a multiple of three, etc.). We apply this convention of considering only syntactically correct inputs to all other problems as well. Let \(A\) be some computational problem and let \(x\) be an instance for \(A\). If we consider an algorithm for \(A\), and input \(x\) is malformed, then we can immediately reject. If we are building a reduction from \(A\) to some problem \(B\), then whenever we hit a malformed input \(x\) then we can output a fixed \(y\) not in \(B\). (In our reductions \(B\) is never \(\Sigma^*\) so this is always possible.)

Copeland\(^\alpha\) elections can often be considered in terms of appropriate graphs. This representation is particularly useful when we face control problems that modify the structure of the candidate set, since in this case operations on some election directly translate into suitable operations on the corresponding graph. For candidate control problems, we—instead of using reductions from X3C—construct reductions from the vertex cover problem.

**Name:** VertexCover.

**Given:** An undirected graph \(G\) and a nonnegative integer \(k\).

**Question:** Is there a set \(W\) such that \(W \subseteq V(G), \|W\| \leq k\), and for every edge \(e = \{u, v\}, e \in E(G)\), it holds that \(e \cap W \neq \emptyset\)?
2.5 Resistance and Vulnerability

Not all election systems can be affected by each control type; if not the system is then said to be immune to this type of control. For example, if a candidate $c$ is a Condorcet winner then it is impossible to prevent him or her from being a Condorcet winner by deleting other candidates (see [BTT92] and [HHR07a] for more such immunity results). However, for Copeland$^\alpha$ elections it is easy to see that for each type of control defined in Section 2.2 there is a scenario where the outcome of the election can indeed be changed via conducting the corresponding control action. If an election system is not immune to some type of control (as witnessed by such a scenario), it is said to be susceptible to this control type.

Proposition 2.3 For each rational number $\alpha$, $0 \leq \alpha \leq 1$, Copeland$^\alpha$ is susceptible to each type of control defined in Section 2.2.

We say that an election system (Copeland$^\alpha$ or Copeland$^\alpha_{\text{Irrational}}$, in our case) is resistant to a particular attack (be it a type of control or of bribery) if the appropriate computational problem is NP-hard (and, for the control case only, if susceptibility holds in addition). On the other hand, if the computational problem is in P (and, in the case of control, if susceptibility holds in addition), then we say the system is vulnerable to this attack. Because of how our bribery and control problems are defined, the vulnerability definition merely requires that there exist a polynomial-time algorithm that determines whether a successful bribe or control action exists on a given input. However, in every single one of our vulnerability proofs we will provide something far stronger. We will provide a polynomial-time algorithm that actually finds and displays a successful bribe or control action on each input for which a successful bribe or control action exists, and on each input where no successful bribe or control action exists will announce that fact.

The notions of resistance and vulnerability (and of immunity and susceptibility) for control problems in election systems were introduced by Bartholdi, Tovey, and Trick [BTT92], and we here follow the definition alteration of [HHR07b] of resistance from “NP-complete” to “NP-hard,” as that change is compelling. However, for all resistance claims in this paper NP-membership is clear, and so NP-completeness in fact does hold.

3 Bribery

In this section we present our results on the complexity of bribery for the Copeland$^\alpha$ election systems, where $\alpha$ is a rational number with $0 \leq \alpha \leq 1$. Our main result, which will be presented in Section 3.1, is that each such system is resistant to bribery, regardless of voters’ rationality and of our mode of operation (constructive versus destructive). In Section 3.2, we will provide vulnerability results for Llull and Copeland$^0$ with respect to “microbribery.”

3.1 Resistance to Bribery

Theorem 3.1 For each rational $\alpha$, $0 \leq \alpha \leq 1$, Copeland$^\alpha$ is resistant to both constructive and destructive bribery in both the rational-voters case and the irrational-voters case, in
both the nonunique-winner model and in the unique-winner model.

We prove Theorem 3.1 via Theorems 3.2 and 3.4 and Corollary 3.5 below. Our proofs employ an approach that we call the UV technique. This technique proceeds by constructing bribery instances where the only briberies that could possibly ensure that our favorite candidate \( p \) is a winner (who, in the destructive case, defeats our despised candidate) involve only voters who rank a group of special candidates (often the group needs contains exactly two candidates, \( u \) and \( v \)), above \( p \). The remaining voters, the bystanders so to speak, can be used to create appropriate padding and structure within the election.

The remainder of this section is devoted to proving Theorem 3.1. We start with the case of rational voters in Theorems 3.2 and 3.4 below and then argue that the analogous results for the case of irrational voters follow via, essentially, the same proof.

**Theorem 3.2** For each rational number \( \alpha \), \( 0 \leq \alpha \leq 1 \), Copeland\( ^{\alpha} \) is resistant to constructive bribery in the unique-winner model and to destructive bribery in the nonunique-winner model.

**Proof.** Fix an arbitrary rational number \( \alpha \) with \( 0 \leq \alpha \leq 1 \). Our proof provides reductions from the X3C problem to, respectively, the unique-winner variant of constructive bribery and to the nonunique-winner variant of destructive bribery. Our reductions will only differ regarding the specification of the goal (i.e., regarding which candidate we attempt to make a unique winner or which candidate we prevent from being a winner) and thus we describe them jointly as, essentially, a single reduction.

Our reduction will produce an instance of an appropriate bribery problem with an odd number of voters and so we will never have ties in head-to-head contests. Thus our proof works regardless of which rational number \( \alpha \) with \( 0 \leq \alpha \leq 1 \) is chosen.

Let \((B, S)\) be an instance of X3C, where \( B = \{b_1, b_2, \ldots, b_{3k}\} \), \( S \) is a collection \( \{S_1, S_2, \ldots, S_n\} \) of three-element subsets of \( B \) with \( \bigcup_{j=1}^{n} S_j = B \), and \( k \geq 1 \). If our input does not meet these conditions then we output a fixed instance of our bribery problem with a negative answer.

Construct a Copeland\( ^{\alpha} \) election \( E = (C, V) \) as follows. The candidate set \( C \) is \( \{u, v, p\} \cup B \), where none of \( u \), \( v \), and \( p \) is in \( B \). The voter set \( V \) contains \( 2n + 4k + 1 \) voters of the following types.

1. For each \( S_i \), we introduce one voter of type (i) and one voter of type (ii):
   
   \[
   \begin{align*}
   (i) & \quad u > v > S_i > p > B - S_i, \\
   (ii) & \quad B - S_i > p > u > S_i. 
   \end{align*}
   \]

2. We introduce \( k \) voters for each of the types (iii)-1, (iii)-2, (iv)-1, and (iv)-2:
   
   \[
   \begin{align*}
   (iii)-1 & \quad u > v > p > B, \\
   (iii)-2 & \quad v > u > p > B, \\
   (iv)-1 & \quad u > B > p > v, \\
   (iv)-2 & \quad v > B > p > u. 
   \end{align*}
   \]
3. We introduce a single type (v) voter:

\[ (v) \quad B > p > u > v. \]

We have the following relative vote-scores:

1. \( vs_E(u, v) = 2n + 1 \geq 2k + 1 \), where the inequality follows from our assumption \( \bigcup_{j=1}^{n} S_j = B \) (which implies \( n \geq \|B\|/3 = k \)),
2. \( vs_E(u, p) = vs_E(v, p) = 2k - 1 \),
3. for each \( i \in \{1, 2, \ldots, 3k\} \), \( vs_E(u, b_i) = vs_E(v, b_i) \geq 2k + 1 \),
4. for each \( i \in \{1, 2, \ldots, 3k\} \), \( vs_E(b_i, p) = 1 \), and
5. for each \( i, j \in \{1, 2, \ldots, 3k\} \) with \( i \neq j \), \( |vs_E(b_i, b_j)| = 1 \).

For example, to see that \( vs_E(u, b_i) \geq 2k + 1 \) for each \( i \in \{1, 2, \ldots, 3k\} \), note that each \( b_i \) is in at least one \( S_j \) because of \( \bigcup_{j=1}^{n} S_j = B \), so the voters of types (i) and (ii) give \( u \) an advantage of at least two votes over \( b_i \). Furthermore, the voters of types (iii)-1, (iii)-2, (iv)-1, and (iv)-2 give \( u \) an advantage of \( 2k \) additional votes over each \( b_i \), and the single type (v) voter gives each \( b_i \) a one-vote advantage over \( u \). Summing up, we obtain \( vs_E(u, b_i) \geq 2 + 2k - 1 = 2k + 1 \). The other relative vote-scores are similarly easy to verify.

These relative vote-scores yield the following Copeland scores or upper bounds on such scores:

1. \( \text{score}^\alpha_E(u) = 3k + 2 \),
2. \( \text{score}^\alpha_E(v) = 3k + 1 \),
3. for each \( i \in \{1, 2, \ldots, 3k\} \), \( \text{score}^\alpha_E(b_i) \leq 3k \),
4. \( \text{score}^\alpha_E(p) = 0 \).

To prove our theorem, we need the following claim.

**Claim 3.3** The following three statements are equivalent:

1. \( (B, S) \in X3C \).
2. Candidate \( u \) can be prevented from winning via bribing at most \( k \) voters of \( E \).
3. Candidate \( p \) can be made a unique winner via bribing at most \( k \) voters of \( E \).

**Proof of Claim 3.3.** (1) implies (2). It is easy to see that if \( (B, S) \in X3C \) then \( u \) can be prevented from being a winner by bribing at most \( k \) voters: It is enough to bribe those type (i) voters that correspond to a cover of size \( k \) to report \( p \) as their top choice (while not changing anything else in their preference lists): \( p > u > v > S_i > B - S_i \). Call the
resulting election $E'$. In $E'$ the following relative vote-scores change: $vs(p, u) = vs(p, v) = n + k - (n - k) - 2k + 1 = 1$ and $vs_{E'}(p, b_i) \geq 1$ for each $i \in \{1, 2, \ldots, 3k\}$, while all other relative vote-scores remain unchanged. Thus $score^v_{E'}(p) = 3k + 2$, $score^v_{E'}(u) = 3k + 1$, $score^v_{E'}(v) = 3k$, and $score^v_{E'}(b_i) < 3k$ for each $i \in \{1, 2, \ldots, 3k\}$, so $p$ defeats all other candidates and is the unique winner. In particular, $u$ has been prevented from winning via bribing at most $k$ voters in $E$.

(2) implies (3). Suppose that $u$ can be prevented from being a winner by bribing at most $k$ voters. Note that $u$ defeats everyone except $p$ by more than $2k$ votes in $E$. This means that via bribery of at most $k$ voters $u$’s score can decrease by at most one. Thus, to prevent $u$ from being a winner via such a bribery, we need to ensure that $u$ receives a Copeland$^\alpha$ score of $3k + 1$ and some candidate other than $u$ gets a Copeland$^\alpha$ score of $3k + 2$, that is, that candidate defeats everyone. Neither $v$ nor any of the $b_i$’s can possibly obtain a Copeland$^\alpha$ score of $3k + 2$ via such a bribery, since bribery of at most $k$ voters can affect only head-to-head contests where the relative vote-scores of the participants are at most $2k$. Thus, via such a bribery, $u$ can be prevented from winning only if $p$ can be made a (in fact, the unique) winner of our election.

(3) implies (1). Let $W$ be a set of at most $k$ voters whose bribery ensures that $p$ is a unique winner of our election. Thus we know that $||W|| = k$ and that $W$ contains only voters who rank both $u$ and $v$ above $p$ (for otherwise $p$ would not defeat both $u$ and $v$), which is the case only for voters of types (i), (iii)-1, and (iii)-2. Furthermore, a bribery that makes $p$ the unique winner has to ensure that $p$ defeats all members of $B$; note that the type (iii)-1 and (iii)-2 voters in $E$ already rank $p$ above all of $B$. Thus, via a simple counting argument, $W$ must contain exactly $k$ type (i) voters that correspond to a size-$k$ cover of $B$.

Since our reduction is computable in polynomial time, Claim 3.3 completes the proof of Theorem 3.2.

\[ \Box \] Claim 3.3

**Theorem 3.4** For each rational $\alpha$, $0 \leq \alpha \leq 1$, Copeland$^\alpha$ is resistant to constructive bribery in the nonunique-winner model and to destructive bribery in the unique-winner model.

**Proof.** Fix an arbitrary rational number $\alpha$ with $0 \leq \alpha \leq 1$. As in the proof of Theorem 3.2, we handle the appropriate constructive and destructive cases jointly using essentially the same reduction for each of them, differing only in the specification of the goal of the briber. Thus we describe our reductions from X3C to the appropriate constructive and destructive bribery problems as a single reduction, separately specifying only the goals for each of the cases.

Our reduction works as follows. We are given an X3C instance $(B, S)$, where $B = \{b_1, b_2, \ldots, b_{3k}\}$ is a set, $S$ is a collection $\{S_1, S_2, \ldots, S_n\}$ of three-element subsets of $B$ with $\bigcup_{j=1}^n S_j = B$, and $k$ is a positive integer. We form an election $E = (C, V)$, where $C = \{p, s, t, u, v\} \cup B$ and where $V$ is as specified below this paragraph. In the nonunique-winner constructive case we want to ensure that $p$ is a winner and in the unique-winner
destructive case we want to prevent $s$ from being the unique winner. In each case we want to achieve our goal via bribing at most $k$ voters from $V$. $V$ contains $2n + 24k + 1$ voters of the following types:

1. For each $S_i$, we introduce one voter of type (i) and one voter of type (ii):
   
   (i) $s > t > u > v > S_i > p > B - S_i$,
   
   (ii) $\overline{B - S_i} > p > v > u > t > s > \overline{S_i}$.

2. We introduce $k$ voters for each of the types (iii)-1, (iii)-2, (iv)-1, and (iv)-2:

   (iii)-1 $s > t > u > v > p > B$,
   
   (iii)-2 $s > t > v > u > p > B$,
   
   (iv)-1 $u > \overline{B} > p > s > v > t$,
   
   (iv)-2 $v > \overline{B} > p > s > u > t$.

3. We introduce $20k$ normalizing voters:

   $2k$ voters of type (v)-1 $u > v > t > p > s > B$,
   
   $2k$ voters of type (v)-2 $u > v > s > t > p > B$,
   
   $3k$ voters of type (v)-3 $s > t > u > v > p > B$,
   
   $3k$ voters of type (v)-4 $s > v > t > u > p > B$,
   
   $3k$ voters of type (v)-5 $\overline{B} > t > p > u > s > v$,
   
   $k$ voters of type (v)-6 $\overline{B} > p > s > u > v > t$,
   
   $3k$ voters of type (v)-7 $\overline{B} > s > p > u > v > t$,
   
   $3k$ voters of type (v)-8 $\overline{B} > p > s > v > t > u$.

4. Finally, we introduce a single type (vi) voter:

   (vi) $B > p > u > v > s > t$.

In the nonunique-winner constructive case we want to ensure that $p$ is a winner and in the unique-winner destructive case we want to prevent $s$ from being the unique winner. In each case we want to achieve our goal via bribing at most $k$ voters. Thus within our bribery we can affect the results of head-to-head contests between only those candidates whose relative vote-scores are, in absolute value, at most $2k$. In $E$, we have the following relative vote-scores:

1. $\text{vs}_E(s,t) > 2k$, $\text{vs}_E(s,u) > 2k$, $\text{vs}_E(s,v) > 2k$, $\text{vs}_E(t,p) > 2k$, $\text{vs}_E(t,u) > 2k$, $\text{vs}_E(v,t) > 2k$, and $\text{vs}_E(u,v) > 2k$,

2. $\text{vs}_E(s,p) = \text{vs}_E(u,p) = \text{vs}_E(v,p) = 2k - 1$,

3. for each $i \in \{1, 2, \ldots, 3k\}$, $\text{vs}_E(b_i,p) = 1$, $\text{vs}_E(s,b_i) > 2k$, $\text{vs}_E(t,b_i) > 2k$, $\text{vs}_E(u,b_i) > 2k$, and $\text{vs}_E(v,b_i) > 2k$, and
4. for each \(i, j \in \{1, 2, \ldots, 3k\} \) with \(i \neq j\), we have \(|vs_E(b_i, b_j)| = 1\).

To analyze \(E\), let \(E'\) denote an arbitrary election resulting from \(E\) via bribing at most \(k\) voters. The relative vote-scores among any two candidates in \(E\) yield the following Copeland\(^a\) scores:

1. \(score_E^a(s) = 3k + 4\), and since we have \(vs_E(s, c) > 2k\) for each candidate \(c \in C\) with \(p \neq c \neq s\), it follows that \(3k + 3 \leq score_E^a(s)\).
2. For each \(x \in \{t, u, v\}\), \(score_E^a(x) = 3k + 2\), and since we have \(vs_E(s, x) > 2k\), \(vs_E(t, u) > 2k\), \(vs_E(u, v) > 2k\), and \(vs_E(v, t) > 2k\), it follows that \(score_E^a(x) \leq 3k + 2\).
3. \(score_E^a(p) = 0\), and since we have \(vs_E(t, p) > 2k\), it follows that \(score_E^a(p) \leq 3k + 3\).
4. For each \(i \in \{1, 2, \ldots, 3k\}\), \(score_E^a(b_i) \leq 3k\), and since we have \(vs_E(x, b_i) > 2k\) for each candidate \(x \in \{s, t, u, v\}\), it follows that \(score_E^a(b_i) \leq 3k\).

Thus \(s\) is the unique winner of \(E\), and the only candidate who is able to prevent \(s\) from being the unique winner via bribing at most \(k\) voters is \(p\).

We claim that \((B, S) \in X3C\) if and only if \(s\) can be prevented from being the unique winner via bribing at most \(k\) voters of \(E\). (Equivalently, \((B, S) \in X3C\) if and only if we can ensure that \(p\) is a winner via at most \(k\) bribes.)

From left to right, if \(S\) has an exact cover for \(B\), then \(s\) can be prevented from being the unique winner by bribing the \(k\) type (i) voters that correspond to this cover of \(B\). In more detail, if the \(k\) bribed voters rank \(p\) on top while leaving their preferences otherwise unchanged (i.e., their votes are now \(p > s > t > u > v > S_i > B - S_i\)), then the only relative vote-scores that have changed in this new election, call it \(E'\), are: \(vs_{E'}(p, s) = vs_{E'}(p, u) = vs_{E'}(t, p) = 4k - 1\), and \(vs_{E'}(p, b_i) = 1\) for each \(i \in \{1, 2, \ldots, 3k\}\). It follows that \(p\) and \(s\) tie for winner in \(E'\) with \(score_{E'}^a(p) = score_{E'}^a(s) = 3k + 3\).

From right to left, suppose that \(s\) can be prevented from being the unique winner via bribing at most \(k\) voters of \(E\). By construction, for each election \(E'\) that results from \(E\) via bribing at most \(k\) voters, this is possible only if \(score_{E'}^a(p) = score_{E'}^a(s) = 3k + 3\). Let \(W\) be a set of at most \(k\) voters whose bribery ensures that \(s\) is not the unique winner in the resulting election. Since \(vs_E(t, p) > 2k\), it is not possible for \(p\) to win the head-to-head contest with \(t\) via such a bribery. Thus, for \(p\) to obtain a score of \(3k + 3\), \(p\) must win the head-to-head contests with each candidate in \(\{s, u, v\} \cup B\). However, since \(vs_E(s, p) = vs_E(u, p) = vs_E(v, p) = 2k - 1\), we have \(|W| = k\) and every voter in \(W\) must rank each of \(s\), \(u\), and \(v\) ahead of \(p\). Thus, \(W\) can contain only voters of types (i), (iii)-1, (iii)-2, (v)-2, (v)-3, and (v)-4. However, since \(p\) also needs to defeat each member of \(B\) and since all voters of types (iii)-1, (iii)-2, (v)-2, (v)-3, and (v)-4 rank \(p\) ahead of each member of \(B\), \(W\) must contain exactly \(k\) type (i) voters that correspond to an exact cover for \(B\).

Since our reduction is computable in polynomial time, this completes the proof of Theorem 3.4. \(\square\)

The proofs of the above theorems have an interesting feature. When we discuss bribery, we never rely on the fact that the voters are rational. Thus we can
simply allow the voters to be irrational and form Copeland\(^\alpha\)\text{Irrational}-bribery and Copeland\(^\alpha\)\text{Irrational}-destructive-bribery instances simply by deriving the voters’ preference tables from the voters’ preference lists given in the above proofs. It is easy to see that the proofs remain valid after this change; the bribes that our proofs use simply require that particular voters (namely the type (i) voters in \(E\)) prefer \(p\) to everyone else. Thus we have the following corollary to the proofs of Theorems 3.2 and 3.4.

**Corollary 3.5** For each rational number \(\alpha, 0 \leq \alpha \leq 1\), Copeland\(^\alpha\) is resistant to both constructive bribery and destructive bribery, both in the nonunique-winner model and in the unique-winner model.

Theorems 3.2 and 3.4 and Corollary 3.5 together constitute a proof of Theorem 3.1 and establish that for each rational \(\alpha, 0 \leq \alpha \leq 1\), Copeland\(^\alpha\) possesses broad—essentially perfect—resistance to bribery regardless of whether we are interested in rational or irrational voters and regardless of whether we are interested in constructive or destructive results. However, the next section shows that this perfect picture is, in fact, only near-perfect when we consider microbribes, which don’t allow changing the complete preferences of voters at once but rather change the results of head-to-head contests between candidates in the voters’ preferences. We will show that there is an efficient way of finding optimal microbribes for the case of irrational voters in Copeland\(^\alpha\) elections.

### 3.2 Vulnerability to Microbribery for Irrational Voters

In this section we explore the problems related to microbribery of irrational voters. In standard bribery problems, which were considered in Section 3.1, we ask whether it is possible to ensure that a designated candidate \(p\) is a winner (or, in the destructive case, to ensure that \(p\) is not a winner) via modifying the preference tables of at most \(k\) voters. That is, we can at unit cost completely redefine the preference table of each voter bribed. Often such an approach is right: We pay for a service (namely, the modification of the reported preference table) and we pay for it in bulk (when we buy a voter, we have secured his or her total obedience). However, sometimes it may be more reasonable to adopt a more local approach and to pay separately for each preference-table entry flip.

Throughout the remainder of this section we will use the term *microbribe* to refer to flipping a single entry in a preference table, and we will use the term *microbribery* to refer to bribing possibly irrational voters via microbribes. Recall that by “irrational voters” we simply mean that they are allowed to have, but not that they each must have, irrational preferences.

For each rational \(\alpha, 0 \leq \alpha \leq 1\), we define the following two problems.

**Name:** Copeland\(^\alpha\)\text{Irrational}-microbribery and Copeland\(^\alpha\)\text{Irrational}-destructive-microbribery.

**Given:** A set \(C\) of candidates, a collection \(V\) of voters specified via their preference tables over \(C\), a distinguished candidate \(p \in C\), and a nonnegative integer \(k\).
**Question (constructive):** Is it possible to ensure that $p$ is a winner of the given election by flipping at most $k$ entries in the preference tables of voters in $V$?

**Question (destructive):** Is it possible to guarantee that $p$ is not a winner of the given election by flipping at most $k$ entries in the preference tables of voters in $V$?

We can flip multiple entries in the preference table of the same voter, but we have to pay separately for each flip. The microbribery problems for Copeland$^\alpha_{\text{Irrational}}$ are very similar in flavor to the so-called bribery problems for approval voting that were studied by Faliszewski, Hemaspaandra, and Hemaspaandra [FHH06a], where unit cost for flipping approvals or disapprovals of voters are paid. However, the proofs for Copeland$^\alpha_{\text{Irrational}}$ seem to be much more involved than their counterparts for approval voting. The reason is that Copeland$^\alpha_{\text{Irrational}}$ elections allow for very subtle and complicated interactions between the candidates’ scores.

Before we proceed with our results, let us define some notation that will be useful throughout this section. Let $E$ be an election with candidate set $C = \{c_1, c_2, \ldots, c_m\}$ and voter collection $V = \{v_1, v_2, \ldots, v_n\}$. We define three functions, $\text{wincost}_E$, $\text{tiecost}_E$, and $\text{cost}_E$, that describe the costs of ensuring a victory or a tie of a given candidate in a particular head-to-head contest.

**Definition 3.6** Let $E = (C, V)$ be an election and let $c_i$ and $c_j$ be two distinct candidates in $C$.

1. By $\text{wincost}_E(c_i, c_j)$ we mean the minimum number of microbribes that ensure that $c_i$ defeats $c_j$ in a head-to-head contest. If $c_i$ already wins this contest then $\text{wincost}_E(c_i, c_j) = 0$.

2. By $\text{tiecost}_E(c_i, c_j)$ we mean the minimum number of microbribes that ensure that $c_i$ ties with $c_j$ in their head-to-head contest, or $\infty$ if $E$ has an odd number of voters and thus ties are impossible.

3. Define $\text{cost}_E(c_i, c_j)$ as $\min(\text{wincost}_E(c_i, c_j), \text{tiecost}_E(c_i, c_j))$.

Within Llull, $\text{cost}_E(c_i, c_j)$ simply gives the minimum number of microbribes that ensures that $c_i$ receives a Llull point from the head-to-head contest between $c_i$ and $c_j$. We often do not care about how the score of $c_j$ changes; rather, we merely want to reach a tie between $c_i$ and $c_j$ if possible, and have $c_i$ defeat $c_j$ otherwise.

Our first result regarding microbribery is that destructive microbribery is easy for Copeland$^\alpha_{\text{Irrational}}$. Since this is the paper’s first vulnerability proof, we take this opportunity to remind the reader (recall Section 2.5) that although the definition of vulnerability requires only that there be a polynomial-time algorithm to determine whether a successful action (in the present case, a destructive microbribe) exists, we will in each vulnerability proof provide something far stronger, namely a polynomial-time algorithm that both determines whether a successful action exists and that, when so, explicitly finds a successful action.
**Theorem 3.7** For each rational $\alpha$, $0 \leq \alpha \leq 1$, Copeland$_{\text{IrRational}}^\alpha$ is vulnerable to destructive microbribery.

**Proof.** Fix an arbitrary rational number $\alpha$ with $0 \leq \alpha \leq 1$. We give an algorithm for Copeland$_{\text{IrRational}}^\alpha$:

Let $E = (C, V)$ be the input election where $C = \{p, c_1, c_2, \ldots, c_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$, and let $k$ be the number of microbriberies that we are allowed to make. We define predicate $M(E, p, c_i, k)$ to be true if and only if there is a bribery of cost at most $k$ that ensures that $c_i$’s score is higher than that of $p$. Our algorithm computes $M(E, p, c_i, k)$ for each $c_i \in C$ and accepts if and only if it is true for at least one of them. We now describe how to compute $M(E, p, c_i, k)$.

We set $E_1$, $E_2$, and $E_3$ to be elections identical to $E$ except that

1. in $E_1$, $p$ defeats $c_i$ in their head-to-head contest,
2. in $E_2$, $p$ loses to $c_i$ in their head-to-head contest,
3. in $E_3$, $p$ ties $c_i$ in their head-to-head contest (we disregard $E_3$ if the number of voters is odd and thus ties are impossible).

Let $k_1$, $k_2$, and $k_3$ be the minimum costs of microbriberies that transform $E$ to $E_1$, to $E_2$, and to $E_3$, respectively. Such microbriberies involve only the head-to-head contest between $p$ and $c_i$. We define predicate $M'(E', p, c_i, k')$, where $E' \in \{E_1, E_2, E_3\}$ and where $k'$ is a nonnegative integer, to be true if and only if there is a microbribery of cost at most $k'$ that does not involve the head-to-head contest between $p$ and $c_i$ but that ensures that $c_i$’s score is higher than $p$’s. It is easy to see that

$$M(E, p, c_i, k) \iff M'(E_1, p, c_i, k - k_1) \lor M'(E_2, p, c_i, k - k_2) \lor M'(E_3, p, c_i, k - k_3).$$

Thus we focus on the problem of computing $M'(E', p, c_i, k')$.

Let $(E', k')$ be one of $(E_1, k - k_1)$, $(E_2, k - k_2)$, and $(E_3, k - k_3)$. We define $\text{promote}_{E'}(c_i, w', w'', t)$, where $c_i \in C$ is a candidate and $w'$, $w''$, and $t$ are nonnegative integers, to be the minimum cost of a microbribery that, when applied to $E'$, increases $c_i$’s Copeland$_{\alpha}$ score by $w' + (1 - \alpha)w'' + at$ via ensuring that

1. $c_i$ wins additional $w'$ head-to-head contests against candidates in $C - \{p\}$ that used to defeat $c_i$ originally,
2. $c_i$ wins additional $w''$ head-to-head contests against candidates in $C - \{p\}$ with whom $c_i$ used to tie originally,
3. $c_i$ ties additional $t$ head-to-head contest with candidates in $C - \{p\}$ that used to defeat $c_i$ originally.

---

5We stress that we have optimized our algorithm for simplicity rather than for performance.
If such a microbribery does not exist then we set \( \text{promote}_{E'}(c_i, w', w'', t) \) to be \( \infty \). It is an easy exercise to see that \( \text{promote}_{E'} \) is computable in polynomial time: One can either use a natural dynamic programming approach or employ a simple greedy strategy.\(^6\)

We define \( \text{demote}_{E'}(c_i, \ell', \ell'', t) \) to be the minimum cost of a microbribery that, when applied to election \( E' \), decreases \( p \)'s score by \( \ell' + \alpha \ell'' + (1 - \alpha) t \) via ensuring that

1. \( p \) loses additional \( \ell' \) head-to-head contests to candidates in \( C - \{c_i\} \), whom \( p \) used to defeat originally,
2. \( p \) loses additional \( \ell'' \) head-to-head contests to candidates in \( C - \{c_i\} \) with whom \( p \) used to tie originally,
3. \( p \) ties additional \( t \) head-to-head contests with candidates in \( C - \{c_i\} \) whom \( p \) used to defeat originally.

If such a microbribery does not exist then we set \( \text{demote}_{E'}(c_i, \ell', \ell'', t) \) to be \( \infty \). Note that \( \text{demote}_{E'} \) can be computed in polynomial time using a similar algorithm as that for \( \text{promote}_{E'} \).

Naturally, the microbriberies used implicitly within \( \text{promote}_{E'}(c_i, w', w'', t) \), \( \text{demote}_{E'}(c_i, \ell', \ell'', t) \), and within transforming \( E \) to \( E' \) are “disjoint,” i.e., they never involve the same pair of candidates. Thus \( M'(E', p, c_i, k') \) is true if and only if there exist nonnegative integers \( w', w'', \ell', \ell'', t', t'' \in \{0, \ldots, m\} \) such that

\[
\text{score}_{E'}^\alpha(c_i) + (w' + \ell' + (1 - \alpha)(t'' + w'') + \alpha(t' + \ell'')) - \text{score}_{E'}^\alpha(p) > 0
\]

and

\[
\text{promote}_{E'}(c_i, w', w'', t') + \text{demote}_{E'}(c_i, \ell', \ell'', t'') \leq k.
\]

There are only polynomially many combinations of \( w', w'', \ell', \ell'', t', t'' \), and we can try them all. Thus we have given a polynomial-time algorithm for \( M'(E', p, c_i, k') \). Via the observations given at the beginning of our proof this implies that \( M(E, p, c_i, k) \) is computable in polynomial time and the proof is complete. \( \square \)

The above algorithm and approach is fairly straightforward. In the destructive case we do not need to worry about any side effects of promoting \( c \) and demoting \( p \). The constructive case is more complicated, but we still are able to obtain polynomial-time algorithms via a fairly involved use of flow networks to model how particular Copeland\(^\alpha\) points travel between candidates. In the remainder of this section we restrict ourselves to the values \( \alpha \in \{0,1\} \), and we remind the reader that Copeland\(^1\) is Liull voting.

A flow network is a network of nodes with directed edges through which we want to transport some amount of goods from the source to the sink (these are two designated nodes). Each edge \( e \) can carry up to \( c(e) \) units of flow, and transporting each unit of flow through \( e \) costs \( a(e) \). In the max-flow problem we ask what is the maximum amount of flow

\(^6\)The dynamic-programming approach would be even more useful if we allowed the voters to have distinct prices for switching preference-table entries.
that can be transported from the source to the sink, ignoring the costs. In the min-cost-flow problem we have a target flow value $F$, less than or equal to the maximum flow value, and the goal is to find a way of transporting $F$ units of flow from the source to the sink, minimizing the cost.

We now define the notions related to flow networks more formally.

**Definition 3.8**

1. A flow network is a quintuple $(K, s, t, c, a)$, where $K$ is a set of nodes that includes the source $s$ and the sink $t$, $c : K^2 \rightarrow \mathbb{N}$ is the capacity function, and $a : K^2 \rightarrow \mathbb{N}$ is the cost function. We assume that $c(n_i, n_i) = a(n_i, n_i) = 0$ for each node $n_i \in K$, and that at most one of $c(n_i, n_j)$ and $c(n_j, n_i)$ is nonzero for each pair of nodes $n_i, n_j \in K$. We also assume that if $c(n_i, n_j) = 0$ then $a(n_i, n_j) = 0$ as well.

2. Given a flow network $(K, s, t, c, a)$, a flow is a function $f : K^2 \rightarrow \mathbb{Z}$ that satisfies the following conditions:
   
   (a) For each $u, v \in K$, we have $f(u, v) \leq c(u, v)$, i.e., capacities limit the flow.
   (b) For each $u, v \in K$, we have $f(u, v) = -f(v, u)$, i.e., the flow is skew-symmetric.
   (c) For each $u \in K - \{s, t\}$, we have $\sum_{v \in K} f(u, v) = 0$, i.e., the flow is conserved in all vertices except the source and the sink.

3. The value of flow $f$ is:
   
   $$\text{flowvalue}(f) = \sum_{v \in K} f(s, v).$$

   The particular flow network we have in mind will always be clear from the context and so we will not indicate it explicitly (we will not write it explicitly as a subscript to the function flowvalue).

4. The cost of flow $f$ is defined as:
   
   $$\text{flowcost}(f) = \sum_{u, v \in K} a(u, v)f(u, v).$$

   That is, we pay the price $a(u, v)$ for each unit of flow that passes from node $u$ to node $v$.

Below we define the min-cost-flow problem, which is well-known from the literature. The definition we employ here is not the most general one. The reader might want to consult, e.g., the monograph by Ahuja, Magnanti, and Orlin [AMO93] for a comprehensive discussion of the problem.

**Definition 3.9** Define the min-cost-flow problem as follows: Given a flow network $N = (K, s, t, c, a)$ and a target flow value $F$, find a flow $f$ that has value $F$ (if one exists) and has minimum cost among all such flows, or otherwise indicate that no such flow $f$ exists.

---

7Note that each flow is fully defined via its nonnegative values. Whenever we speak of a flow (e.g., when defining some particular flows) we will speak of its nonnegative part.
The min-cost-flow problem has a polynomial-time algorithm.\footnote{Max-flow and min-cost-flow problems are often defined in terms of capacity and cost functions that are not necessarily limited to nonnegative integer values and so the corresponding flows are not restricted to integer values either. However, crucially for us, it is known that if the capacity and cost functions have integral values (as we have assumed) then there exist optimal solutions to both the max-flow problem and the min-cost-flow problem that use only integer-valued flows and that can be found in polynomial time.} There is a tremendous body of work devoted to flow problems and we will not even attempt to provide a complete list of references here. Instead, we point the reader to the excellent monograph by Ahuja, Magnanti, and Orlin [AMO93], which provides descriptions of polynomial-time algorithms, theoretical analysis, and numerous references to previous work on flow-related problems. We also mention that the issue of flows is so prevalent in the study of algorithms that the textbook of Cormen et al. [CLRS01] contains an exposition of both the max-flow problem and the min-cost-flow problem.

Coming back to the study of constructive microbribery for Llull and Copeland$^0$, we now present the following result.

**Theorem 3.10** Both Llull and Copeland$^0$ are vulnerable to constructive microbribery.

We prove Theorem 3.10 via Lemmas 3.11 through 3.17 below, which cover three cases: (a) an odd number of voters, where all Copeland$^\alpha$ elections with $0 \leq \alpha \leq 1$ are identical due to the lack of ties, (b) Llull with an even number of voters, and (c) Copeland$^0$ with an even number of voters.

**Lemma 3.11** For each rational $\alpha$ with $0 \leq \alpha \leq 1$, there is a polynomial-time algorithm that, given that the number of voters is odd, solves the constructive microbribery problem for Copeland$^\alpha$.

**Proof.** Our input is the nonnegative integer $k$ (the budget) and an election $E = (C, V)$, where the candidate set $C$ is $\{c_0, c_1, \ldots, c_m\}$, the number of voters is odd, and $p = c_0$ is the candidate whose victory we want to ensure via at most $k$ microbribes. Note that we interchangeably use $p$ and $c_0$ to refer to the same candidate; it is sometimes convenient to be able to speak of $p$ and all other candidates uniformly. Since the number of voters is odd, ties never occur. Thus any candidate $c_i$ has the same Copeland$^\alpha$ score for each rational value of $\alpha$, $0 \leq \alpha \leq 1$. Fix an arbitrary such $\alpha$.

We give a polynomial-time algorithm for the constructive microbribery problem. A high-level overview is that we try to find a threshold value $T$ such that there is a microbribery of cost at most $k$ that transforms $E$ into $E'$ that (a) $p$ has score$^\alpha_{E'}$ exactly $T$, and (b) every other candidate has score$^\alpha_{E'}$ at most $T$.

Let $B$ be a number that is greater than the cost of any possible microbribery within $E$ (e.g., $B = \|V\| \cdot \|C\|^2 + 1$). For each possible threshold $T$, we consider a min-cost-flow instance $I(T)$ with nodes $K = C \cup \{s, t\}$, where $s$ is the source and $t$ is the sink, the edge capacities and costs are specified in Figure 1, and the target flow value is

\[
F = \sum_{c_i \in C} \text{score}^\alpha_{E}(c_i) = \frac{\|C\|((\|C\| - 1)}{2}.
\]
\[
e = (s, c_i), \quad c(e) = \text{score}_E^\alpha(c_i), \quad a(e) = 0
\]

\[
e = (c_i, c_j), \quad c(e) = 1, \quad a(e) = \text{wincost}_E(c_j, c_i)
\]

\[
e = (c_0, t) \quad c(e) = T, \quad a(e) = 0
\]

\[
e = (c_i, t), \quad c(e) = T, \quad a(e) = B
\]

\[
e = (c_0, t), \quad c(e) = 0, \quad a(e) = 0
\]

Figure 1: Edge capacities and costs for min-cost-flow instance \(I(T)\), built from election \(E\). Functions \(c\) and \(a\) represent edges capacities and costs, respectively.

<table>
<thead>
<tr>
<th>Edge</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e = (s, c_i)), where (c_i \in C)</td>
<td>(c(e) = \text{score}_E^\alpha(c_i)), (a(e) = 0)</td>
</tr>
<tr>
<td>(e = (c_i, c_j)), where (c_i, c_j \in C) and (\text{vs}_E(c_i, c_j) &gt; 0)</td>
<td>(c(e) = 1), (a(e) = \text{wincost}_E(c_j, c_i))</td>
</tr>
<tr>
<td>(e = (c_0, t))</td>
<td>(c(e) = T), (a(e) = 0)</td>
</tr>
<tr>
<td>(e = (c_i, t)), where (i &gt; 0) and (c_i \in C)</td>
<td>(c(e) = T), (a(e) = B)</td>
</tr>
<tr>
<td>every other edge (e)</td>
<td>(c(e) = 0), (a(e) = 0)</td>
</tr>
</tbody>
</table>

Table: vs\(_E(c_i, c_j)\) and \(c_0, c_1, c_2, c_3\)

<table>
<thead>
<tr>
<th>vs(_E(c_i, c_j))</th>
<th>(c_0)</th>
<th>(c_1)</th>
<th>(c_2)</th>
<th>(c_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c_0)</td>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>(c_1)</td>
<td>-1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>(c_2)</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(c_3)</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 2: A sample election \(E\) for Example 3.12 in the proof of Lemma 3.11.

**Example 3.12** For illustration, consider the following example. Suppose the given election \(E\) has four candidates and three voters, and the preference tables of the voters (who each happen to be rational in this example) can be obtained from their preference orders that are shown in Figure 2, which also gives the corresponding values of \(\text{vs}_E(c_i, c_j)\) for each pair of candidates. Thus we have \(\text{score}_E^\alpha(c_0) = 2\), \(\text{score}_E^\alpha(c_1) = 0\), \(\text{score}_E^\alpha(c_2) = 3\), and \(\text{score}_E^\alpha(c_3) = 1\). Suppose further that we are allowed to do one microbribe, so \(k = 1\). Clearly, one microbribe that changes the preference of the third voter from \(c_2 > c_0\) to \(c_0 > c_2\) will flip the outcome of their head-to-head contest from \(c_2\) winning to \(c_0\) winning, which is enough to reach our goal of making \(c_0\) win the election, and this is of course the cheapest possible successful microbribe. Finally, note that in this example we have \(B = 49\).

For any threshold \(T\) with \(0 \leq T \leq 3\), the flow network \(I(T)\) corresponding to this instance \((E, c_0, k)\) of the constructive microbribery problem is shown in Figure 3, and we have a target flow value of \(F = 6\). Every edge \(e\) in this flow network is labeled by the pair \((c(e), a(e))\) of numbers that give the capacity and the cost of edge \(e\), respectively.

To continue the proof of Lemma 3.11, note that with an odd number of voters, constructive microbribery in Copeland\(^\alpha\) simply requires us to choose for which pairs of
distinct candidates we want to flip the outcome of their head-to-head contest in order to ensure \( p \)'s victory. Thus it is sufficient to represent a microbribery \( M \) as a collection of pairs \((c_i, c_j)\) of distinct candidates for whom we need to flip the result of their head-to-head contest from \( c_i \) winning to \( c_j \) winning. Clearly, given such a collection \( M \), the cheapest way to implement it costs

\[
\sum_{(c_i, c_j) \in M} \text{wincost}_{E}(c_j, c_i).
\]

A crucial observation for our algorithm is that we can directly translate flows to microbribes using the following interpretation. Let \( f \) be a flow of value \( F \) within instance \( I(T) \). The units of flow that travel through the network correspond to Copeland\( ^\alpha \) points. For each \( c_i \in C \), we interpret the amount of flow that goes directly from \( s \) to \( c_i \) as the number of Copeland\( ^\alpha \) points that \( c_i \) has before any microbribery is attempted,\(^9\) and the amount of flow that goes directly from \( c_i \) to \( t \) as the number of Copeland\( ^\alpha \) points that \( c_i \) has after the microbribery (defined by the flow). The units of flow that travel between distinct \( c_i \)'s (i.e., through edges of the form \((c_i, c_j), i \neq j\)) correspond to the microbribes exerted: A unit of flow traveling from node \( c_i \) to \( c_j \) corresponds to changing the result of the head-to-head contest between \( c_i \) and \( c_j \) from \( c_i \) winning to \( c_j \) winning. In this case, the Copeland\( ^\alpha \) point moves from \( c_i \) to \( c_j \) and the cost of the flow increases by \( a(c_i, c_j) = \text{wincost}_{E}(c_j, c_i) \), exactly the minimum cost of a microbribery that flips this contest’s result. Let \( M \) be the microbribery defined, as just described, by flow \( f \). It is easy to see that

\[
\text{flowcost}(f) = B \cdot (F - f(c_0, t)) + \sum_{(c_i, c_j) \in M} \text{wincost}_{E}(c_j, c_i).
\]

\(^9\)Note that for each \( c_i \in C \) any flow of value \( F \) within \( I(T) \) needs to send exactly \( \text{score}^\alpha_E(c_i) \) units from \( s \) to \( c_i \).
procedure Copeland$^\alpha$-odd-microbribery($E = (C, V), k, p$)
begin
    if $p$ is a winner of $E$ then accept;
    $F = \sum_{c_i \in C} \text{score}^E_{\alpha}(c_i) = \|C\|((\|C\|-1)/2);
    \text{for } T = 0 \text{ to } \|C\| - 1 \text{ do}
    begin
        build an instance $I(T)$ of min-cost-flow;
        if $I(T)$ has no flow of value $F$ then
            restart the for loop with the next value of $T$;
        $f$ = a minimum-cost flow for $I(T)$ with integral values only;
        if $f(c_0, t) < T$ then restart the loop;
        $\kappa = \text{flowcost}(f) - B \cdot (F - T)$;
        if $\kappa \leq k$ then accept;
    end;
    reject;
end

Figure 4: The constructive microbribery algorithm for Copeland$^\alpha$ elections with an odd number of voters.

Thus we can easily extract the cost of microbribery $M$ from the cost of flow $f$.

Our algorithm crucially depends on this one-to-one correspondence between flows and microbriberies. (Also, in the proofs of Lemmas 3.14 and 3.17 that cover the case of an even number of voters we simply show how to modify the instances $I(T)$ to handle ties, and we show correspondences between the new networks and microbriberies; the rest of these proofs is the same as here.)

Note that for small values of $T$ no flow of value $F$ exists for $I(T)$. The reason for this is that the edges coming into the sink $t$ might simply not have enough capacity so as to hold a flow of value $F$. Such a situation means that it simply is impossible to guarantee that every candidate gets at most $T$ points; i.e., that there are too many Copeland$^\alpha$ points to distribute.

Figure 4 gives our algorithm for constructive microbribery in Copeland$^\alpha$. This algorithm runs in polynomial time because, as it is well known, the min-cost-flow problem is solvable in polynomial time.

Let us now prove that this algorithm is correct. We have presented above how a flow $f$ of value $F$ within the flow network $I(T)$ (with $0 \leq T \leq F$) defines a microbribery. Based on this, it is clear that if our algorithm accepts then there is a microbribery of cost at most $k$ that ensures $p$’s victory.

On the other hand, suppose now that there exists a microbribery of cost at most $k$ that ensures $p$’s victory in the election. We will show that our algorithm accepts in this case.
Let $M$ be a minimum-cost bribery (of cost at most $k$) that ensures $p$'s victory. As pointed out above, $M$ can be represented as a collection of pairs $(c_i, c_j)$ of distinct candidates for whom we flip the result of the head-to-head contest from $c_i$ winning to $c_j$ winning. The cost of $M$ is

$$\sum_{(c_i, c_j) \in M} \text{wincost}_E(c_j, c_i).$$

Since applying microbribery $M$ ensures that $p$ is a winner, we have that each candidate among $c_1, c_2, \ldots, c_m$ has at most as many Copeland$^\alpha$ points as $p$ does. Let $E'$ be the election that results from $E$ after applying microbribery $M$ to $E$ (i.e., after flipping the results of the contests specified by $M$ in an optimal way, as given by $\text{wincost}_E$). Let $T'$ be $\text{score}_{E'}^\alpha(p)$, $p$'s Copeland$^\alpha$ score after implementing $M$. Clearly, $0 \leq T' \leq \|C\| - 1$.

Consider instance $I(T')$ and let $f_M$ be the flow that corresponds to the microbribery $M$. In this flow each edge of the form $(s, c_i)$ carries flow of its maximum capacity, $\text{score}_E^\alpha(c_i)$, each edge of the form $(c_i, c_j)$ carries units of flow that correspond to exactly the flips of head-to-head contests incurred by $M$ (e.g., an edge $e = (c_i, c_j)$ with $c(e) = 1$ carries one unit of flow exactly if $e$ is listed in $M$, and carries zero units of flow otherwise), and each edge of the form $(c_i, t)$ carries $\text{score}_E^\alpha(c_i)$ units of flow. It is easy to see that this is a legal flow. The cost of $f_M$ is

$$\text{flowcost}(f_M) = B \cdot (F - T') + \sum_{(c_i, c_j) \in M} \text{wincost}_E(c_j, c_i).$$

After applying $M$, $p$ gets $T'$ Copeland$^\alpha$ points that travel to the sink $t$ via edge $(c_0, t)$ with cost $a(c_0, t) = 0$, whereas all the remaining $F - T'$ points travel via edges $(c_i, t)$, $i \in \{1, 2, \ldots, m\}$, with cost $a(c_i, t) = B$. The remaining part of $\text{flowcost}(f_M)$ is the cost of the units of flow traveling through the edges $(c_i, c_j)$ that directly correspond to the cost of microbribery $M$.

Now consider some minimum-cost flow $f_{\text{min}}$ for $I(T')$. Since $f_M$ exists, a minimum-cost flow must exist as well. Clearly, we have

$$\text{flowcost}(f_{\text{min}}) \leq \text{flowcost}(f_M).$$

Let $T''$ be the number of units of flow that $f_{\text{min}}$ assigns to travel over the edge $(c_0, t)$, i.e., $T'' = f_{\text{min}}(c_0, t)$. The only edges with nonzero cost for sending flow through them are those in the set $\{(c_i, c_j) \mid c_i, c_j \in C \land \text{vs}_{E}(c_i, c_j) > 0\} \cup \{(c_i, t) \mid i \in \{1, \ldots, m\}\}$ and thus the cost of $f_{\text{min}}$ can be expressed as (recall that $\text{vs}_{E}(c_i, c_j) > 0$ implies $i \neq j$)

$$\text{flowcost}(f_{\text{min}}) = B \cdot (F - T'') + \sum_{c_i, c_j \in C \land \text{vs}_{E}(c_i, c_j) > 0} f_{\text{min}}(c_i, c_j) \cdot \text{wincost}_E(c_j, c_i).$$

Since $B > \sum_{i, j, i \neq j} \text{wincost}_E(c_i, c_j)$ and for each $c_i, c_j \in C$ such that $\text{vs}_{E}(c_i, c_j) > 0$, we have $f_{\text{min}}(c_i, c_j) \in \{0, 1\}$, and $\text{flowcost}(f_{\text{min}}) \leq \text{flowcost}(f_M)$ it must be that $T'' = T'$.

$$\sum_{c_i, c_j \in C \land \text{vs}_{E}(c_i, c_j) > 0} f_{\text{min}}(c_i, c_j) \cdot \text{wincost}_E(c_j, c_i) \leq \sum_{(c_i, c_j) \in M} \text{wincost}_E(c_j, c_i).$$

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Thus flow $f_{\text{min}}$ corresponds to a microbribery that guarantees $p$’s victory and has cost at most as high as that of $M$. Since $M$ was chosen to have minimum cost among all such microbribries, flow $f_{\text{min}}$ corresponds to a microbribery of minimum cost and our algorithm correctly accepts within the “for loop,” at the latest when reaching $T = T'$.

We now turn to the algorithms showing that Llull and Copeland$^0$ are vulnerable to constructive microbribery when the number of voters is even. In this case we need to take into account that it sometimes is more desirable to have some candidates tie each other in a head-to-head contest than to have one of them win the contest.

**Lemma 3.13** Let $E = (C, V)$ be an election with candidate set $C = \{c_0, c_1, \ldots, c_m\}$ and with an even number of voters, specified via irrational preference tables over $C$. If the election is conducted using Copeland$^0$ then no minimum-cost microbribery that ensures $c_0$’s victory involves either flipping a result of a head-to-head contest between any two distinct candidates $c_i, c_j \in C - \{c_0\}$ from $c_i$ winning to $c_j$ winning or changing a result of a head-to-head contest between these $c_i$ and $c_j$ from a tie to one of them winning.

**Proof.** Our proof follows by way of contradiction. Let $E = (C, V)$ be an election as specified in the lemma. For the sake of a contradiction suppose there is a minimum-cost microbribery $M$ that makes $c_0$ win and that there are two distinct candidates, $c_i$ and $c_j$, in $C - \{c_0\}$ such that microbribery $M$ involves switching the result of the head-to-head contest between these candidates from $c_i$ winning to $c_j$ winning or from a tie to one of them winning. Consider the microbribery $M'$ that is identical to $M$, except that it makes $c_i$ tie with $c_j$ in a head-to-head contest, either via an appropriate number of microbribes if $c_i$ and $c_j$ do not tie originally, or via leaving the corresponding preference-table entries untouched if they do tie initially. Clearly, this microbribery $M'$ has a lower cost than $M$ and it still ensures $c_0$’s victory. A contradiction.

With Lemma 3.13 at hand, we can show that microbribery is easy for Copeland$^0$ for the case of an even number of voters.

**Lemma 3.14** There is a polynomial-time algorithm that solves the constructive microbribery problem for Copeland$^0$, given that the number of voters is even.

**Proof.** Our input is election $E = (C, V)$ where $C = \{c_0, c_1, \ldots, c_m\}$, $p = c_0$, and $V$ is a collection of an even number of voters, each specified via an irrational preference table over $C$. Our algorithm is essentially the same as that used in the proof of Lemma 3.11, except that instead of using instances $I(T)$ we now use instances $J(T)$ defined below. In this proof we show how to construct these instances and how they correspond to microbribries within $E$. The proof of Lemma 3.11 shows how to use such a correspondence to solve the microbribery problem at hand.
Let $T$ be a nonnegative integer, $0 \leq T \leq \|C\| - 1$. Instance $J(T)$ is somewhat different from the $I(T)$ used in the proof of Lemma 3.11. In particular, due to Lemma 3.13, we only model microbriberies that have the following effects on our election:

1. For each two candidates $c_i, c_j$ in $C - \{c_0\}$ the result of the head-to-head contest between $c_i$ and $c_j$ may possibly turn to be a tie.

2. For each candidate $c_i$ in $C - \{c_0\}$ the result of a head-to-head contest between $c_0$ and $c_i$ may possibly turn to either a tie (from $c_i$ defeating $c_0$) or to $c_0$ defeating $c_i$ (from either a tie or from $c_i$ defeating $c_0$).

Our instance $J(T)$ contains special nodes, sets $C'$ and $C''$ below, to handle these possible interactions. We define

$$C' = \{c_{ij} \mid i, j \in \{1, 2, \ldots, m\} \land \text{vs}_E(c_i, c_j) > 0\}, \text{ and}$$

$$C'' = \{c_i \mid i \in \{1, 2, \ldots, m\} \land \text{vs}_E(c_i, c_0) \geq 0\}.$$

We define $J(T)$ to be a flow network with nodes $K = C \cup C' \cup C'' \cup \{s, t\}$, where $s$ is the source and $t$ is the sink and the edge capacities and costs are as stated in Figure 5. (As before, we set $B$ be a number that is greater than the cost of any possible microbribery within $E$, e.g., $B = \|V\| \cdot \|C\|^2 + 1$.) The target flow value is

$$F = \sum_{v \in K} c(s, v).$$

Instance $J(T)$ is fairly complicated but it in fact does follow the instance of microbribery that we have at hand closely. As in the case of an odd number of voters, the units of flow that travel through the network are interpreted as Copeland$^0$ points, and appropriate flows are interpreted as specifying microbriberies.

**Claim 3.15** Each flow $f$ of value $F$ that travels through the network $J(T)$ corresponds to a microbribery within $E$ that gives each candidate $c_i \in C$ exactly $f(c_i, t)$ Copeland$^0$ points.

**Proof of Claim 3.15.** The following observations will prove the claim.

1. For each $c_i \in C$, the units of flow that enter $c_i$ from $s$ correspond to the number of Copeland points $c_i$ has in $E$, prior to any microbribery.

2. For each $c_i \in C$, the units of flow that go directly from $c_i$ to $t$ correspond to the number of Copeland points $c_i$ has after a microbribery as specified by the flow.

3. For each pair of distinct candidates $c_i, c_j \in C - \{c_0\}$ such that $c_i$ defeats $c_j$ in their head-to-head contest in $E$, we have an edge $e = (c_i, c_j)$ with capacity 1 and cost $\text{tiecost}_E(c_{ij})$. A unit of flow that travels through $e$ corresponds to microbriberies that make $c_i$ tie with $c_j$: $c_i$ loses the point, we pay the cost $\text{tiecost}_E(c_j, c_i)$, and then the unit of flow goes directly to $t$. (From Lemma 3.13 we know that we do not need to handle any other possible interactions between $c_i$ and $c_j$ in their head-to-head contest.)
### Parameters

<table>
<thead>
<tr>
<th>Edge</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e = (s, c_i)$, where $c_i \in C$</td>
<td>$c(e) = \text{score}_E^i(c_i)$ $a(e) = 0$</td>
</tr>
<tr>
<td>$e = (c_i, c_{ij})$, where $c_i \in C - {c_0}$ and $c_{ij} \in C'$</td>
<td>$c(e) = 1$ $a(e) = \text{tiecost}<em>E(c</em>{ij}, c_i)$</td>
</tr>
<tr>
<td>$e = (c_i, t)$, where $c_i \in C - {c_0}$</td>
<td>$c(e) = T$ $a(e) = B$</td>
</tr>
<tr>
<td>$e = (c_{ij}, t)$, where $c_{ij} \in C'$</td>
<td>$c(e) = 1$ $a(e) = B$</td>
</tr>
<tr>
<td>$e = (c_0, c_{i0})$, where $c_{i0} \in C''$ and $\text{vs}<em>E(c_i, c</em>{i0}) &gt; 0$</td>
<td>$c(e) = 1$ $a(e) = \text{tiecost}<em>E(c</em>{0}, c_i)$</td>
</tr>
<tr>
<td>$e = (s, c_{i0})$, where $c_{i0} \in C''$ and $\text{vs}<em>E(c_i, c</em>{i0}) = 0$</td>
<td>$c(e) = 1$ $a(e) = 0$</td>
</tr>
<tr>
<td>$e = (c_{i0}, t)$, where $c_{i0} \in C''$</td>
<td>$c(e) = 1$ $a(e) = B$</td>
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</tr>
<tr>
<td>every other edge $e$</td>
<td>$c(e) = 0$ $a(e) = 0$</td>
</tr>
</tbody>
</table>

Figure 5: Edge capacities and costs for min-cost-flow instance $J(T)$, built from election $E$.

4. For each candidate $c_i \in C - \{c_0\}$ such that $c_i$ defeats $c_0$ in a head-to-head contest, we need to allow for the possibility that a microbribery causes $c_0$ to either tie with $c_i$ or to defeat $c_i$. Fix an arbitrary such $c_i$. A unit of flow that travels directly from $c_i$ to $c_{i0}$ and then directly to $t$ corresponds to a microbribery after which $c_0$ ties with $c_i$: $c_i$ loses the point but $c_0$ does not receive it and the cost of the flow increases by $\text{tiecost}_E(c_{0}, c_i)$.

On the other hand, if that unit of flow travels from $c_i$ to $c_{i0}$ and then directly to $c_0$, then this corresponds to a microbribery after which $c_0$ defeats $c_i$. The point travels from $c_i$ to $c_0$ and the cost of the flow increases by $\text{wincost}_E(c_{0}, c_i)$.

If there is no flow entering node $c_{i0}$ then this means that our microbribery does not change the result of a head-to-head contest between $c_0$ and $c_i$.

5. For each candidate $c_i \in C - \{c_0\}$ such that $c_i$ ties with $c_0$ in a head-to-head contest before any microbribery is attempted, we need to allow for $c_0$ defeating $c_i$ after the microbribery. Fix any such $c_i$. A unit of flow that travels directly from $s$ to $c_{i0}$ and then directly to $c_0$ corresponds to a microbribery after which $c_0$ defeats $c_i$ in a
head-to-head contest: $c_0$ gets an additional point and the cost of the flow increases by $\text{wincost}_E(c_0, c_i) - \text{tiedist}_E(c_0, c_i) = \text{wincost}_E(c_0, c_i)$.

On the other hand, a unit of flow that travels from $s$ directly to $c_0$ and then directly to $t$ corresponds to a microbribery that does not change the result of a head-to-head contest between $c_0$ and $c_i$.

Based on the above comments, the claim is proven. \hfill \Box \quad \text{Claim 3.15}

To continue the proof of Lemma 3.14, let $M_f$ be a microbribery defined by flow $f$ of value $F$ within $J(T)$, $0 \leq T \leq ||C|| - 1$, assuming that one exists. Let $\text{cost}(M_f)$ be the minimum cost of implementing microbribery $M_f$. A close inspection of instance $J(T)$ gives that the cost of such a flow $f$ is

$$\text{flowcost}(f) = B \cdot (F - f(c_0, t)) + \text{cost}(M_f).$$

By the arguments presented in the proof of Lemma 3.11, if for a given $J(T)$, $0 \leq T \leq ||C|| - 1$, there exists a flow of value $F$ then a minimum-cost flow $f_{\text{min}}$ of value $F$ corresponds to a minimum-cost microbribery that ensures that $c_0$ receives $T$ points and each other candidate receives at most $T$ points. Thus to solve the constructive microbribery problem for Copeland$^0$ elections with an even number of voters it is enough to run the algorithm from Figure 4, using the instances $J(T)$ instead of $I(T)$ and using the new value of $F$. \hfill \Box

Let us now turn to the case of constructive microbribery within Llull. The following lemma reduces the set of microbriberies we need to model in this case.

**Lemma 3.16** Let $E = (C, V)$ be an election with candidate set $C = \{c_0, c_1, \ldots, c_m\}$ and with an even number of voters, specified via irrational preference tables over $C$. If the election is conducted using Llull then no minimum-cost microbribery that ensures $c_0$’s victory involves obtaining a tie in a head-to-head contest between any two distinct candidates in $C - \{c_0\}$.

**Proof.** Our proof is again by way of contradiction. Let $E = (C, V)$ be an election as specified in the lemma. Suppose there is a minimum-cost microbribery that ensures $c_0$’s victory and that involves obtaining a tie in a head-to-head contest between two distinct candidates in $C - \{c_0\}$, say $c_i$ and $c_j$. That is, before this microbribery we have that either $c_i$ defeats $c_j$ or $c_j$ defeats $c_i$ in their head-to-head contest but afterward they are tied. Clearly, a microbribery that is identical to this one, except that it does not change the result of the head-to-head contest between $c_i$ and $c_j$ (i.e., one that does not microbribe any voters to flip their preference-table entries regarding $c_i$ and $c_j$) has a smaller cost and still ensures $c_0$’s victory. \hfill \Box

**Lemma 3.17** There is a polynomial-time algorithm that solves the constructive microbribery problem for Llull, given that the number of voters is even.

---

10The only effect of not reaching a tie in the head-to-head contest between $c_i$ and $c_j$ is that now one of them has one point less.
Proof. We give a polynomial-time algorithm for constructive microbribery in Llull elections with an even number of voters. Our input is a budget \( k \in \mathbb{N} \) and an election \( E = (C, V) \) where \( C = \{c_0, c_1, \ldots, c_m\} \), \( p = c_0 \), and where \( V \) contains an even number of voters specified via their irrational preference tables over \( C \). Our goal is to ensure \( p \)'s victory via at most \( k \) microbribes.

We use essentially the same algorithm as that given in the proof of Lemma 3.11, except that instead of using instances \( I(T) \) we now employ instances \( L(T) \) that are designed to handle tie issues as appropriate for Llull. Lemma 3.16 tells us that our min-cost-flow instances \( L(T) \) do not need to model microbriberies that incur ties between pairs of candidates in \( C - \{c_0\} \). However, we do need to model possible microbribery-induced ties between \( c_0 \) and each \( c_i \) in \( C - \{c_0\} \).

Define \( B \) to be a number that is greater than the cost of any possible microbribery within \( E \) (e.g., \( B = \| V \| \cdot \| C \|^2 + 1 \) will do). Further, define the following three sets of nodes:

\[
C' = \{c'_i \mid c_i \in C\},
C'' = \{c_{ij} \mid i < j \land c_i, c_j \in C \land \text{vs}_E(c_i, c_j) = 0\}, \text{ and}
C''' = \{c_{0i} \mid c_i \in C \land \text{vs}_E(c_i, c_0) > 0\}.
\]

Our flow network of \( L(T) \) has nodes \( K = C \cup C' \cup C'' \cup C''' \cup \{s, t\} \), where \( s \) is the source and \( t \) is the sink and the capacities and costs of edges are defined in Figure 6. Each instance \( L(T) \) asks for a minimum-cost flow of value

\[
F = \sum_{c_i \in C} c(s, c_i).
\]

A flow \( f \) of value \( F \) within \( L(T) \), \( 0 \leq T \leq \| C \| - 1 \), corresponds to a microbribery \( M_f \) within election \( E \) that leaves each candidate \( c_i \) with exactly \( f(c_i, c'_i) \) Llull points. Similarly as in the proofs of Lemmas 3.11 and 3.14, points traveling through the network of \( L(T) \) are here interpreted as Llull points.

Claim 3.18 Each flow \( f \) of value \( F \) corresponds to a microbribery \( M_f \) within \( E \) that gives each candidate \( c_i \) exactly \( f(c_i, c'_i) \) points.

Proof of Claim 3.18. The following observations will prove the claim.

1. For each \( c_i \in C \), the units of flow that enter \( c_i \) from \( s \) are interpreted as the Llull points that \( c_i \) has before any microbribery is attempted.

2. For each \( c_i \in C \), the units of flow that travel directly from \( c_i \) to \( c'_i \) are interpreted as the Llull points that \( c_i \) has after the microbribery defined by \( f \) has been performed. A certain subtlety occurs if a candidate \( c_i \in C - \{c_0\} \) originally defeats \( c_0 \) but our flow models a microbribery in which \( c_0 \) and \( c_i \) end up tied in their head-to-head contest. Then we have a single Llull point that travels from \( c_i \) to \( c'_i \), then to \( c_0 \) through \( C_{0i} \), at
Figure 6: Edge capacities and costs for min-cost-flow instance $L(T)$, built from election $E$.

<table>
<thead>
<tr>
<th>Edge</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e = (s, c_i)$, where $c_i \in C$</td>
<td>$c(e) = score^1_E(c_i)$, $a(e) = 0$</td>
</tr>
<tr>
<td>$e = (c_i, c_j)$, where $c_i, c_j \in C - {c_0}$ and $vs_E(c_i, c_j) &gt; 0$</td>
<td>$c(e) = 1$, $a(e) = wincost_E(c_j, c_i)$</td>
</tr>
<tr>
<td>$e = (c_i, c_{ij})$, where $i &lt; j$, $c_i, c_j \in C$ and $vs_E(c_i, c_j) = 0$</td>
<td>$c(e) = 1$, $a(e) = wincost_E(c_j, c_i)$</td>
</tr>
<tr>
<td>$e = (c_j, c_{ij})$, where $i &lt; j$, $c_i, c_j \in C$ and $vs_E(c_i, c_j) = 0$</td>
<td>$c(e) = 1$, $a(e) = wincost_E(c_i, c_j)$</td>
</tr>
<tr>
<td>$e = (c_{ij}, t)$, where $i &lt; j$, $c_i, c_j \in C$ and $vs_E(c_i, c_j) = 0$</td>
<td>$c(e) = 1$, $a(e) = B$</td>
</tr>
<tr>
<td>$e = (c_i, c_0i)$, where $c_i \in C - {c_0}$ and $vs_E(c_i, c_0) &gt; 0$</td>
<td>$c(e) = 1$, $a(e) = wincost(c_0, c_i)$</td>
</tr>
<tr>
<td>$e = (c_i, c_0i)$, where $c_i \in C - {c_0}$ and $vs_E(c_i, c_0) &gt; 0$</td>
<td>$c(e) = 1$, $a(e) = tiecost(c_0, c_i)$</td>
</tr>
<tr>
<td>$e = (c_0i, c_0)$, where $c_i \in C - {c_0}$ and $vs_E(c_i, c_0) &gt; 0$</td>
<td>$c(e) = 1$, $a(e) = 0$</td>
</tr>
<tr>
<td>$e = (c_i, c'_i)$, where $c_i \in C$</td>
<td>$c(e) = T$, $a(e) = 0$</td>
</tr>
<tr>
<td>$e = (c'_i, t)$, where $c_i \in C - {c_0}$</td>
<td>$c(e) = T$, $a(e) = B$</td>
</tr>
<tr>
<td>$e = (c'_0, t)$</td>
<td>$c(e) = T$, $a(e) = 0$</td>
</tr>
<tr>
<td>every other edge $e$</td>
<td>$c(e) = 0$, $a(e) = 0$</td>
</tr>
</tbody>
</table>

cost $tiecost_E(c_0, c_i)$, and then to $c'_0$. This way the same unit of flow is accounted both for the score of $c_0$ and for the score of $c_i$. Note that such a unit of flow then travels to $t$ through edge $(c_0, t)$ at zero cost.

3. For any two distinct candidates $c_i$ and $c_j$ such that $c_i, c_j \in C - \{c_0\}$ where $c_i$ defeats $c_j$ in a head-to-head contest in $E$, a unit of flow traveling from $c_i$ to $c_j$ corresponds to a microbribery that flips the result of their head-to-head contest. Thus $c_j$ receives the Llull point and the cost of the flow increases by $wincost_E(c_j, c_i)$.

4. For any $c_i \in C - \{c_0\}$ where $c_i$ defeats $c_0$ in a head-to-head contest in $E$, a unit of flow traveling from $c_i$ to $c_0i$ to $c_0$ corresponds to a microbribery that flips the result of their head-to-head contest. Thus $c_i$ receives the Llull point and the cost of the flow increases by $wincost_E(c_0, c_i)$. Note that since the edge $(c_0i, c_0)$ has capacity 1 we enforce that at most one unit of flow travels from $c_i$ to $c_0$ (either on a path $c_i, c_0i, c_0$...
(modeling a microbribery that flips the result of the head-to-head contest between \(c_i\) and \(c_0\) to \(c_0\) winning) or on a path \(c_i, c'_i, c_0, c_0\) (modeling a microbribery that enforces a tie between \(c_0\) and \(c_i\)).

5. For each \(c_i, c_j \in C\), we have to take into account the possible that \(c_i\) and \(c_j\) are tied in their head-to-head contest within \(E\), but via microbribery we want to change the result of this contest. Let \(c_i, c_j \in C\) be two such candidates and let \(i < j\). Here, a unit of flow traveling from \(c_i\) to \(c_{ij}\) (or, analogously, from \(c_j\) to \(c_{ij}\)) is interpreted as a microbribery that ensures \(c_j\)’s (\(c_i\)’s) victory in the head-to-head contest. Since \(c_i\) and \(c_j\) were already tied, \(c_j\) (\(c_i\)) already has his or her point for the victory and \(c_i\) (\(c_j\)) gets rid of his or her point through the node \(c_{ij}\). The cost of the flow increases by \(\text{wincost}_E(c_j, c_i)\) (respectively, by \(\text{wincost}_E(c_i, c_j)\)). Also, we point out that via the introduction of node \(c_{ij}\) we enforce that only one of \(c_i\) and \(c_j\), let us call him or her \(c_k\), can lose a point by sending it from \(c_k\) to \(c_{ij}\) and then to \(t\); the capacity of edge \((c_{ij}, t)\) is only one.

The above comments prove the claim. \(\square\)  

Claim 3.18

To continue the proof of Lemma 3.17, let us now analyze the cost of \(f\). It is easy to see that each unit of flow that is not accounted for as a Llull point of \(c_0\) reaches the sink \(t\) via an edge of cost \(B\). Also, the only other edges through which units of flow travel that have nonzero costs are those that define the microbribery \(M_f\). Thus the cost of our flow \(f\) can be expressed as

\[
\text{flowcost}(f) = B \cdot (F - f(c_0, c'_0)) + \text{cost}(M_f).
\]

Fix \(T\) such that \(0 \leq T \leq \|C\| - 1\). Given the above properties of \(L(T)\) and by the arguments presented in the proof of Lemma 3.11, if a flow of value \(F\) exists within the flow network of instance \(L(T)\), then it corresponds to a microbribery that ensures that \(c_0\) has exactly \(T\) Llull points and every other candidate has at most \(T\) Llull points. Thus if there exists a value \(T'\) such that

1. there is a flow of value \(F\) in \(L(T')\) and
2. the cost of a minimum-cost flow \(f\) of value \(F\) in \(L(T')\) is \(K\),

then there is a microbribery of cost \(K - B \cdot (F - T')\) that ensures \(c_0\)’s victory.

On the other hand, via Lemma 3.16 and our correspondence between flows for \(L(T)\) and microbriberies in \(E\), there is a value \(T''\) such that a minimum-cost flow in \(L(T'')\) corresponds to a minimum-cost microbribery that ensures \(p\)’s victory. Thus the algorithm from Figure 4, used with the instances \(L(T)\) instead of \(I(T)\) and with our new value of \(F\), solves the constructive microbribery problem for Llull with an even number of voters in polynomial time. \(\square\)

Together, Theorem 3.7 and Lemmas 3.11, 3.14, and 3.17 show that both Llull and Copeland\(^0\), for the case of irrational voters, are vulnerable to microbribery, both in the
constructive and in the destructive settings. It is interesting to note that all our microbribery proofs above would work just as well if we considered a slight twist to the definition of the microbribery problem, namely, if instead of saying that each flip in a voter’s preference table has unit cost we would assume that each voter has potentially different prices for flipping the entries in his or her preference table. The reason for this is that the functions \( wincost \), \( tiecost \), and \( cost \) would still be easily computable and our algorithms do not depend on the values of those functions being polynomially bounded.

An interesting direction for further study of the complexity of bribery within Llull and Copeland\(^\alpha \) systems is to consider a version of the microbribery problem for the case of rational voters. There, one would pay unit cost for a switch of two adjacent candidates on a given voter’s preference list. Naturally, we would also like to know what is the exact complexity of microbribery for Copeland\(^\alpha \) when \( \alpha \) is a rational number between 0 and 1.

4 Control

In this section we focus on the complexity of control in Copeland\(^\alpha \) elections. Intuitively, in control problems we are trying to ensure that our preferred candidate \( p \) is a winner (or, in the destructive case, that our despised candidate is not a winner) of a given election via affecting this election’s structure (e.g., via adding, deleting, or partitioning either candidates or voters). As opposed to the bribery problems, in control problems we are never allowed to change any of the votes and, consequently, the issues that we encounter regarding control problems and the proof techniques we use are quite different from those presented in the previous section.

Somewhat surprisingly, the literature regarding the complexity of control problems is quite scarce. To the best of our knowledge, the only election systems for which a comprehensive analysis was conducted are plurality, Condorcet, and approval voting (see [BTT92, HHR07a, HHR07b]; see also [PRZ07] for some results on (variants of) approval voting, single nontransferable vote, and cumulative voting with respect to constructive control via adding voters). Among plurality, Condorcet, and approval voting, plurality appears to be the least vulnerable to control and so it is natural to compare our new results with those for plurality. Our main result in this section is Theorem 4.1 below.

**Theorem 4.1** Let \( \alpha \) be a rational number with \( 0 \leq \alpha \leq 1 \). Copeland\(^\alpha \) elections are resistant and vulnerable to control types as indicated in Table 1. The same results hold for the case of irrational voters and in both the nonunique-winner model and the unique-winner model.

In particular, we will prove in this section that the notion widely referred to in the literature simply as “Copeland elections,” which we here for clarity call Copeland\(^0\), possesses all ten of our basic types (see Table 1) of constructive resistance (and in addition, even has constructive AC\(u\) resistance). And we will establish that the other notion that in the literature is occasionally referred to as “Copeland elections,” namely Copeland\(^0\), as well
as Llull elections, which are here denoted by Copeland\(^1\), both possess all ten of our basic types of constructive resistance. However, we will show that Copeland\(^0\) and Copeland\(^1\) are vulnerable to this eleventh type of constructive control, the incongruous but historically resonant notion of constructive control by adding an unlimited number of candidates (i.e., CCAC\(_u\)).

Note that Copeland\(^0\)\(^.5\) has a higher number of constructive resistances, by three, than even plurality, which was before this paper the reigning champ. (Although the results regarding plurality in Table 1 are stated for the unique-winner version of control, for all the table’s Copeland\(^\alpha\) cases, \(0 \leq \alpha \leq 1\), our results hold both in the cases of unique winners and of nonunique winners, thus allowing an apples-to-apples comparison to hold.) Admittedly, plurality does perform better with respect to destructive candidate control problems, but still our study of Copeland\(^\alpha\) makes significant steps forward in the quest for a fully control-resistant natural election system with an easy winner problem.

This section is organized as follows. The next two subsections are devoted to proving Theorem 4.1, and the concluding subsection considers the case of control in elections with a bounded number of candidates. Section 4.1 focuses on the first part of Table 1 and studies control problems that affect the candidate structure. Section 4.2 is devoted to voter control and covers the second part of Table 1. Finally, in Section 4.3 we take the role of someone who tries to solve the in-general-resistant control problems and we devise some efficient

<table>
<thead>
<tr>
<th>Control type</th>
<th>Copeland(^\alpha)</th>
<th>Plurality</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(\alpha = 0)</td>
<td>(0 &lt; \alpha &lt; 1)</td>
</tr>
<tr>
<td>Control type</td>
<td>CC</td>
<td>DC</td>
</tr>
<tr>
<td>AC(_u)</td>
<td>V</td>
<td>V</td>
</tr>
<tr>
<td>AC</td>
<td>R</td>
<td>V</td>
</tr>
<tr>
<td>DC</td>
<td>R</td>
<td>V</td>
</tr>
<tr>
<td>RPC-TP</td>
<td>R</td>
<td>V</td>
</tr>
<tr>
<td>RPC-TE</td>
<td>R</td>
<td>V</td>
</tr>
<tr>
<td>PC-TP</td>
<td>R</td>
<td>V</td>
</tr>
<tr>
<td>PC-TE</td>
<td>R</td>
<td>V</td>
</tr>
<tr>
<td>PV-TE</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>PV-TP</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>AV</td>
<td>R</td>
<td>R</td>
</tr>
<tr>
<td>DV</td>
<td>R</td>
<td>R</td>
</tr>
</tbody>
</table>

Table 1: Comparison of control results for Copeland\(^\alpha\) elections, where \(\alpha\) with \(0 \leq \alpha \leq 1\) is a rational number, and for plurality-rule elections. R means resistance to a particular control type and V means vulnerability. The results regarding plurality are due to Bartholdi, Tovey, and Trick [BTT92] and Hemaspaandra, Hemaspaandra, and Rothe [HHR07a]. (Note that CCAC and CCDC resistance results for plurality, not handled explicitly in [BTT92, HHR07a], follow immediately from the respective CCAC\(_u\) and DCAC\(_u\) results.)
algorithms for the case where the number of candidates is bounded.

All our resistance results regarding candidate control follow via reductions from vertex cover and all our vulnerability results follow via greedy algorithms. Our resistance results for the case of control by modifying voter structure follow from reductions from the X3C problem.

4.1 Candidate Control

We start our discussion of candidate control within Copeland$^\alpha$ with the results on destructive control. It is somewhat disappointing that for each rational $\alpha$, $0 \leq \alpha \leq 1$, Copeland$^\alpha$ is vulnerable to each type of destructive candidate control. On the positive side, our vulnerability proofs follow via natural greedy algorithms and will allow us to smoothly get into the spirit of candidate-control problem.

4.1.1 Destructive Candidate Control

The results for destructive control by adding and deleting candidates use the following observation.

Observation 4.2 Let $(C, V)$ be an election, and let $\alpha$ be a fixed rational number such that $0 \leq \alpha \leq 1$. For every candidate $c \in C$ it holds that:

$$score^\alpha_{(C,V)}(c) = \sum_{d \in C \setminus \{c\}} score^\alpha_{(\{c,d\},V)}(c).$$

Theorem 4.3 For each rational number $\alpha$ with $0 \leq \alpha \leq 1$, Copeland$^\alpha$ is vulnerable to destructive control via adding candidates (both limited and unlimited, i.e., DCAC and DCAC$^u$), in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.

Proof. Our input is a set $C$ of candidates, a set $D$ of spoiler candidates, a collection $V$ of voters with preferences (either preference lists or preference tables) over $C \cup D$, a candidate $p \in C$, and a nonnegative integer $k$ (for the unlimited version of the problem we let $k = \|D\|$). We ask if there is a subset $D'$ of $D$ such that $\|D'\| \leq k$ and $p$ is not a winner (is not a unique winner) of Copeland$^\alpha$ election $E' = (C \cup D', V)$. Note that if $k = 0$, this amounts to determining whether $p$ is not a winner (is not a unique winner) of election $E$, which can easily be done in polynomial time.

For the remainder of this proof we will assume that $k > 0$. Let $c$ be a candidate in $(C \cup D) \setminus \{p\}$. We define $a(c)$ to be the maximum value of the expression

$$score^\alpha_{(C \cup D', V)}(c) - score^\alpha_{(C \cup D', V)}(p)$$

under the condition that $D' \subseteq D$, $c \in C \cup D'$ and $\|D'\| \leq k$. From Observation 4.2, it follows that $a(c)$ is the maximum value of

$$score^\alpha_{(C \cup \{c\}, V)}(c) - score^\alpha_{(C \cup \{c\}, V)}(p) + \sum_{d \in D' \setminus \{c\}} \left(score^\alpha_{(\{c,d\}, V)}(c) - score^\alpha_{(\{p,d\}, V)}(p)\right)$$

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under the condition that \( D' \subseteq D \), \( c \in C \cup D' \) and \( \|D'\| \leq k \).

Clearly, \( p \) can be prevented from being a winner (the unique winner) if and only if there exists a candidate \( c \in (C \cup D) - \{p\} \) such that \( a(c) > 0 \) (\( a(c) \geq 0 \)).

Given a candidate \( c \), it is easy to construct in polynomial time a set \( D' \) that yields the value \( a(c) \). We start with \( D' = \emptyset \). If \( c \in D \), we add \( c \) to \( D' \). Then we add those candidates \( d \in D - D' \) to \( D' \) such that \( \alpha(c,d) - \alpha(p,d) \) is maximal and positive, until \( \|D'\| = k \) or until no more such candidates exist.

**Theorem 4.4** For each rational number \( \alpha \) with \( 0 \leq \alpha \leq 1 \), Copeland\( ^\alpha \) is vulnerable to destructive control via deleting candidates (DCDC), in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.

**Proof.** Our approach is similar to that of the previous theorem. We are given an election \( E = (C,V) \), a candidate \( p \in C \), and a nonnegative integer \( k \). Our goal is to check if \( p \) can be prevented from being a winner (the unique winner) via deleting at most \( k \) candidates in \( C - \{p\} \).

Let \( c \) be a candidate in \( C - \{p\} \). We define \( d(c) \) to be the maximum value of the expression

\[
\alpha(c,D,V) - \alpha(p,D,V)
\]

under the condition that \( D \subseteq C \), \( p,c \notin D \) and \( \|D\| \leq k \).

From Observation 4.2, it follows that \( d(c) \) is the maximum value of

\[
\alpha(c,V) - \alpha(p,V) + \sum_{d \in D} \left( \alpha(c,d,V) - \alpha(p,d,V) \right)
\]

under the condition that \( D \subseteq C \), \( p,c \notin D \) and \( \|D\| \leq k \).

Clearly, \( p \) can be prevented from being a winner (the unique winner) if and only if there exists a candidate \( c \in C - \{p\} \) such that \( d(c) > 0 \) (\( d(c) \geq 0 \)).

Given a candidate \( c \in C - \{p\} \), it is easy to construct in polynomial time a set \( D \) that yields the value \( d(c) \). Simply let \( D \) consist of as many as possible but at most \( k \) candidates \( d \notin \{c,p\} \) such that \( \alpha(c,d,V) - \alpha(p,d,V) \) is maximal and positive.

Destructive control via partitioning of candidates (with or without run-off) is also easy.

**Theorem 4.5** For each rational number \( \alpha \) with \( 0 \leq \alpha \leq 1 \), Copeland\( ^\alpha \) is vulnerable to destructive control via partitioning of candidates and via partitioning of candidates with run-off (in both the TP and TE model, i.e., DCPC-TP, DCPC-TE, DCRPC-TP, and DCRPC-TE), in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.

**Proof.** It is easy to see that in the TP model, \( p \) can be prevented from being a winner via partitioning of candidates (with or without run-off) if and only if there is a set \( C' \subseteq C \)
such that \( p \in C' \) and \( p \) is not a winner of \((C', V)\). It follows that \( p \) can be prevented from being a winner if and only if \( p \) can be prevented from being a winner by deleting at most \(|C| - 1\) candidates, which can be determined in polynomial time by Theorem 4.4.

For the case of the TE model, it is easy to see that if there is a set \( C' \subseteq C \) such that \( p \in C' \) and \( p \) is not a unique winner of \((C', V)\) then \( p \) can be prevented from being a unique winner via partitioning of candidates (with or without run-off). One simply partitions the candidates into \( C' \) and \( C - C' \) and thus \( p \) fails to advance to the final stage. On the other hand, if \( p \) can be prevented from being a winner (a unique winner) via partitioning of candidates (with or without run-off) in the TE model, then there exists a set \( C' \subseteq C \) such that \( C' \in p \) and \( p \) is not a unique winner of \((C', V)\). This is so because then either \( p \) does not advance to the final stage (and this means that \( p \) is not a unique winner of his or her first-stage election) or \( p \) is not a winner (not a unique winner) of the final stage (note that not being a winner implies not being a unique winner).

Thus, \( p \) can be prevented from being a winner (a unique winner) via partitioning of candidates (with or without run-off) in the TE model if and only if there is a set \( C' \subseteq C \) such that \( p \in C' \) and \( p \) is not a unique winner of \((C', V)\). Clearly, such a set exists if and only if \( p \) can be prevented from being a unique winner via deleting at most \(|C| - 1\) candidates, which can be tested in polynomial time.

It remains to show that Copeland\(^\alpha\) is vulnerable to destructive control via partitioning of candidates (with or without run-off), both in the rational and irrational voter model, in the unique-winner model in the TP model. In the argument below we focus on the DCRPC-TP case but it is easy to see that essentially the same reasoning works for DCPC-TP.

First we determine if \( p \) can be precluded from being a winner in our current control scenario. This can be done in polynomial time as explained above. If \( p \) can be precluded from being a winner, \( p \) can certainly be precluded from being a unique winner, and we are done. For the remainder of the proof, suppose that \( p \) cannot be precluded from being a winner in our current control scenario, i.e., for every set \( D \subseteq C \) such that \( p \in D \), \( p \) is a winner of \((D, V)\). Let

\[
D_1 = \{ c \in C - \{p\} \mid p \text{ beats } c \text{ in a head-to-head contest} \}
\]

and let \( D_2 = D - (D_1 \cup \{p\}) \). Note that for all \( c \in D_2 \), \( p \) ties \( c \) in a head-to-head contest, since otherwise \( p \) would not be a winner of \((\{c, p\}, V)\). If \( D_2 = \emptyset \), then \( p \) is a Condorcet winner and no partition (with or without run-off) can prevent \( p \) from being a unique winner [HHR07a]. For the remainder of the proof, we assume that \( D_2 \neq \emptyset \). We will show that \( p \) can be precluded from being the unique winner in our current control scenario.

If \( \alpha < 1 \), we let the first subelection be \((\{p\} \cup D_1, V)\). Note that \( p \) is the unique winner of this subelection. The final stage of the election involves \( p \) and one or more candidates from \( D_2 \). Note that every pair of candidates in \( D_2 \cup \{p\} \) is tied in a head-to-head election (for if \( c \) would beat \( d \) in a head-to-head election, \( c \) would be the unique winner of \((\{c, d, p\}, V)\), which contradicts the assumption that \( p \) is a winner of every subelection it participates in).

It follows that all candidates that participate in the final stage of the election are winners, and so \( p \) is not the unique winner.
Finally, consider the case that \( \alpha = 1 \). Then \( \text{score}_{(C,V)}(p) = \|C\| - 1 \). If there is a candidate \( d \in C - \{p\} \) such that \( \text{score}_{(C,V)}(d) = \|C\| - 1 \), then \( d \) will always (i.e., in every subelection containing \( d \)) be a winner, and thus \( p \) will not be a unique winner of the final stage of the election, regardless of which partition of \( C \) was chosen. Now suppose that \( \text{score}_{(C,V)}(d) < \|C\| - 1 \) for all \( d \in C - \{p\} \). Then \( \text{score}_{(C,V)}(d) \leq \|C\| - 2 \) for all \( d \in C - \{p\} \).

Let \( c \) be a candidate in \( D_2 \) and let the first subelection be \( (C - \{c\}, V) \). Let \( C' \) be the set of winners of \( (C - \{c\}, V) \). Since \( \text{score}_{(C - \{c\}, V)}(p) = \|C\| - 2 \), it holds that \( p \in C' \) and for every \( d \in C' - \{p\} \), \( \text{score}_{(C - \{c\}, V)}(d) = \|C\| - 2 \). Since \( \text{score}_{(C,V)}(d) \leq \|C\| - 2 \), it follows that \( c \) beats \( d \) in a head-to-head election. The final stage of the election involves candidates \( C' \cup \{c\} \). Note that \( \text{score}_{(C' \cup \{c\}, V)}(c) = \|C'\| \), and thus \( c \) is a winner of the election, and we have precluded \( p \) from being a unique winner. \( \square \)

The above vulnerability results for the case of destructive candidate control should be contrasted with essentially perfect resistance to constructive candidate control (with the exception of the unlimited version of control via adding candidates) that is shown in the next section.

### 4.1.2 Constructing Instances of Elections

Many of our proofs in the next section require constructing fairly involved instances of Copeland\(^a\) elections. In this section we provide several lemmas and observations that simplify building such instances.

We first note that each election \( E = (C, V) \) induces a directed graph \( G(E) \) whose vertices are \( E \)'s candidates and whose edges correspond to the results of head-to-head contests in \( E \). That is, for each two distinct vertices of \( G(E) \) (i.e., for each two candidates) \( a \) and \( b \) there is an edge from \( a \) to \( b \) if and only if \( a \) defeats \( b \) in their head-to-head contest (i.e., if and only if \( v_{SE}(a,b) > 0 \)). Naturally, \( G(E) \) does not depend on the value of \( \alpha \). The following fundamental result is due to McGarvey.

**Lemma 4.6 ([McG53])** There is a polynomial-time algorithm that given as input an antisymmetric directed graph \( G \) outputs an election \( E \) such that \( G = G(E) \).

**Proof.** For the sake of completeness, we give a sketch of the algorithm. Let \( G \) be an antisymmetric directed graph. We let \( E = (C, V) \) be an election where \( C = V(G) \) and for each edge \((a, b)\) in \( G \) there are exactly two voters, one with preference list \( a > b > C - \{a, b\} \) and one with preference list \( C - \{a, b\} > a > b \). Since \( G \) is antisymmetric, it is easy to see that \( G = G(E) \). \( \square \)

The above basic construction of McGarvey was improved upon by Stearns [Ste59]. While McGarvey’s construction requires twice as many voters as there are edges in \( G \), the construction of Stearns needs at most \( \|V(G)\| + 2 \) voters. Stearns also provides a lower bound on the number of voters that are needed to represent an arbitrary graph via an
election. (It is easy to see that any such graph can be modeled via two irrational voters but the lower bound for the case of rational votes is slightly less trivial.)

We will often construct complicated elections via combining together simpler ones. Whenever we speak of combining two elections, say $E_1 = (C_1, V_1)$ and $E_2 = (C_2, V_2)$, we mean building, via the algorithm from Lemma 4.6, an election $E = (C, V)$ whose election graph is a disjoint union of the election graphs of $E_1$ and $E_2$ with, possibly, some edges added between the vertices of $G(E_1)$ and $G(E_2)$ (in each case we will explicitly state which edges, if any, are added). In particular, we will often want to add some padding candidates to an election, without affecting the original election much. In order to do so, we will typically combine our main election with one of the following, padding, ones.

**Lemma 4.7** Let $\alpha$ be a rational number such that $0 \leq \alpha \leq 1$. For each positive integer $n$ there is an election $\text{Pad}_n = (C, V)$ such that $\|C\| = 2n + 1$ and for each candidate $c_i \in C$ it holds that $\text{score}^\alpha_{\text{Pad}_n}(c) = n$.

**Proof.** Fix a positive integer $n$. By Lemma 4.6 it is enough to show a directed, antisymmetric graph $G$ with $2n + 1$ vertices, each with its indegree and outdegree equal to $n$. We set $G$’s vertex set to be $\{0, 1, \ldots, 2n\}$ and we put an edge from vertex $i$ to vertex $j$ ($i \neq j$) if and only if $(j - i) \mod 2n \leq n$. As a result there is exactly one directed edge between any two distinct vertices and for each vertex $i$ we have edges going out from $i$ to exactly the vertices $(i + 1) \mod 2n, (i + 2) \mod 2n, \ldots, (i + n) \mod n$. Thus, both the indegree and the outdegree of each vertex is equal to $n$ and the proof is complete. \[\Box\]

Lemma 4.6 is very useful when building an election in which we need direct control over the results of all head-to-head contests. However, in many cases specifying explicitly the results of all head-to-head contests would be very tedious. Instead it would be easier to specify the results of only the important head-to-head contests and require all candidates to have particular scores. In the next lemma we show how to construct elections specified in such a way via combining a “small” election containing the important head-to-head contest with a “large” padding election.

**Lemma 4.8** Let $E = (C, V)$ be an election where $C = \{c_1, \ldots, c_n\}$, and let $\alpha$ be a rational number such that $0 \leq \alpha \leq 1$. For each candidate $c_i$ we denote the number of head-to-head ties of $c_i$ in $E$ by $t_i$. Let $k_1, \ldots, k_n$ be a sequence of $n$ nonnegative integers such that for each $k_i$ we have $0 \leq k_i \leq n$. There is an algorithm that in polynomial time outputs an election $E' = (C', V')$ such that:

1. $C' = C \cup D$, where $D = \{d_1, \ldots, d_{2n^2}\}$;
2. for each $i$, $1 \leq i \leq n$, $\text{score}^\alpha_{E'}(c_i) = 2n^2 - k_i + \alpha t_i$;
3. for each $i$, $1 \leq i \leq 2n^2$, $\text{score}^\alpha_{E'}(d_i) \leq n^2 + 1$. 

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Proof. We build $E'$ via combining $E$ with a padding election $F$ (see Lemma 4.7 and the paragraph just before it). $F = (D, W)$, $D = \{d_1, \ldots, d_{2n^2}\}$, is essentially the election Pad$_n^2$ with one arbitrary candidate removed. We partition the candidates in $D$ into $n$ groups, $D_1, \ldots, D_n$, each with exactly $2n$ candidates and we set the results of head-to-head contests between each $c_i \in C$ and the candidates in $D_i$ according to the following scheme. For each $j \in \{1, \ldots, n\}$ such that $i \neq j$, $c_i$ defeats all members of $D_j$ and $c_i$ defeats exactly as many candidates in $D_i$ (and loses to all the remaining ones) as to ensure that
\[
\text{score}_{E'}^{\alpha}(c_i) = 2n^2 - k_i + t_i \alpha.
\]
It is easy to see that this is possible. Including the points that $c_i$ obtains from defeating-or-tying candidates in $C$ and defeating the candidates in $D - D_i$, $c_i$’s score is between $2n^2 - 2n + t_i \alpha$ and $2n^2 - 2n + n - 1 + t_i \alpha$ (note that ties can only happen within $E$). There are $2n$ candidates in $D_i$ and so $c_i$ can reach any score of the form $2n^2 - k + t_i \alpha$, where $k$ is an integer between 0 and $n$, via defeating in head-to-head contests an appropriate number of candidates in $D_i$ and losing to all the remaining ones.

Finally, since $F$ is Pad$_n^2$ with one candidate removed, each $d_i$ gets at most $n^2$ points from defeating other members of $D$ and at most one point from possibly defeating some member of $C$. Thus, for each $d_i \in D$, it holds that $\text{score}_{E'}^{\alpha}(d_i) \leq n^2 + 1$. This completes the proof. 

Instead of invoking Lemma 4.8 directly we will often simply describe an election in terms of the results of important head-to-head contests and the scores of the important candidates and then mention that such an election can be built, with possibly adding extra padding candidates that do not affect the general structure of the election, using Lemma 4.8. In each such a case it will be clear that Lemma 4.8 can in deed be used to build the election we describe.

4.1.3 Constructive Candidate Control

Let us now turn to the case of constructive candidate control. Here we show that resistance holds for Copeland$^a$ in all cases (i.e., for all rational values of $\alpha$ with $0 \leq \alpha \leq 1$ and for all constructive candidate control scenarios) except for CCAC$_u$ for $\alpha \in \{0, 1\}$ where vulnerability holds (see Theorem 4.11).

All our resistance proofs in this section follow via reductions from vertex cover problem. Recall that in vertex cover problem our input is $(G, k)$ where $G$ is an undirected graph with $V(G) = \{1, \ldots, n\}$ and $E(G) = \{e_1, \ldots, e_m\}$ and $k$ is the upper bound on the size of the vertex cover that we seek. Note that if either $m = 0$, $n = 0$, or $k \geq \min(n, m)$ then the instance has a trivial solution and so in our proofs we will always assume that both $n$ and $m$ are nonzero and that $k$ is less than $\min(n, m)$. In each case, if the input to our reduction does not meet these requirements (or is otherwise malformed) the reduction outputs a fixed “yes” instance or a fixed “no” instance depending on the (easily obtained) solution to $(G, k)$ or malformedness of the input.
We use the following notational convention: We identify sets of candidates with elections limited to those candidates. Since in the following proofs we will often speak of various subelections with differing candidate sets but with the same, fixed voter set, this convention simplifies our notation and saves us from explicitly introducing symbols for those subelections.

**Theorem 4.9** Let \( \alpha \) be a rational number such that \( 0 \leq \alpha \leq 1 \). Copeland\(^\alpha\) is resistant to constructive control via adding candidates (CCAC), in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.

**Proof.** We give a reduction from the vertex cover problem. Let \((G, k)\) be an instance of the vertex cover problem, where \(G\) is an undirected graph and \(k\) is the size of the vertex cover that we seek. Let \(\{1, \ldots, n\}\) be the set of \(G\)'s vertices and let \(\{e_1, \ldots, e_m\}\) be the set of its edges. We construct an instance of CCAC for Copeland\(^\alpha\) such that a designated candidate \(p\) can become a winner after adding at most \(k\) candidates if and only if \(G\) has a cover of size at most \(k\). We assume that both \(n\) and \(m\) are nonzero (otherwise we can easily test if \(G\) has a vertex cover of size \(k\) and output a fixed yes or no instance of CCAC).

Our reduction works as follows. Via Lemma 4.8, we build an election \(E' = (C', V')\) such that:

1. \(\|C'\| = 2\ell^2 + \ell\), where \(\ell = 2n + 2m\);
2. \(\{p, e_1, \ldots, e_m\} \subseteq C'\);
3. \(\text{score}^{\alpha}_{E'}(p) = 2\ell^2 - 1\) in the nonunique-winner case (\(\text{score}^{\alpha}_{E'}(p) = 2\ell^2\) in the unique-winner case);
4. for each \(e_i \in C'\), \(\text{score}^{\alpha}_{E'}(e_i) = 2\ell^2\);
5. the scores of all candidates in \(C'\) other than \(p, e_1, \ldots, e_m\) are at most \(2\ell^2 - n - 2\).

We form election \(E = (C, V)\) via combining \(E'\) with candidates \(D = \{1, \ldots, n\}\) (corresponding to the vertices of \(G\)). The results of the head-to-head contests within \(D\) are set arbitrarily, and the head-to-head contests between the members of \(C\) and the members of \(D\) are set as follows: All candidates in \(C - \{e_1, \ldots, e_m\}\) defeat all members of \(D\), and for each \(i \in D\) and each \(e_j \in \{e_1, \ldots, e_m\}\), candidate \(i\) defeats \(e_j\) if \(e_j\) is an edge incident to \(i\) and loses otherwise. Our reduction outputs an instance \((C, D, V, p, k)\) of CCAC that asks if it is possible to choose a subset \(D' \subseteq D\) such that \(\|D'\| \leq k\) and \(p\) is a winner of Copeland\(^\alpha\) election \((C \cup D', V)\). It is clear that this reduction is computable in polynomial time. We will now show that it is correct.

If \(G\) does have a cover of size at most \(k\) then simply add the candidates in \(D\) that correspond to the cover. Adding these candidates increases the score of \(p\) by \(k\), while the scores of the \(e_i\)'s can increase only by \(k - 1\) each, since each edge is incident with at least

\[\text{(11) Strictly speaking, we would consider the preference lists (or tables) of the voters to be restricted to the candidates we are interested in at each given time.}\]
one member of the cover. Thus, \( p \) becomes a winner (the unique winner). For the converse assume that \( p \) can become a winner via adding at most \( k \) candidates from the set \( D \). In order for \( p \) to become a winner (the unique winner), it must be the case that via adding candidates each \( e_i \) gets at least one point less than \( p \). However, this is possible only if we add candidates that correspond to a cover.

Interestingly, when the parameter \( \alpha \) is strictly between 0 and 1 (i.e., \( 0 < \alpha < 1 \)) then Copeland\(^\alpha\) is resistant to constructive control via adding candidates even if we allow adding an unlimited number of candidates (the CCAC\(_u\) case). The reason for this is that for each rational \( \alpha \) strictly between 0 and 1 our construction will enforce, merely via its structure, that we can add at most \( k \) candidates. On the other hand, much to one’s surprise, both Copeland\(^0\) and Copeland\(^1\) are vulnerable to constructive control via adding an unlimited number of candidates (CCAC\(_u\), see Theorem 4.11).

**Theorem 4.10** Let \( \alpha \) be a rational number such that \( 0 < \alpha < 1 \). Copeland\(^\alpha\) is resistant to constructive control via adding an unlimited number of candidates (CCAC\(_u\)), in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.

**Proof.** We give a reduction from the vertex cover problem. Our reduction follows the same general structure as that in the proof of Theorem 4.9.

For the unique-winner case, we will need to specify one of the candidates’ score in terms of a number \( \epsilon \) such that \( 1 - \epsilon \geq \alpha \). Let \( t_1 \) and \( t_2 \) be two positive integers such that \( \alpha = \frac{t_1}{t_2} \) and such that their greatest common divisor is 1. Clearly, two such numbers exist because \( \alpha \) is rational and greater than 0. We set \( \epsilon \) to be \( \frac{1}{t_2} \). By elementary number theoretical arguments, there are two positive integer constants, \( k_1 \) and \( k_2 \), such that \( k_1 \alpha = k_2 - \epsilon \).

Let \((G, k)\) be an instance of the vertex cover problem, where \( G \) is an undirected graph and \( k \) is the bound on the size of the vertex cover that we seek. Let \( \{e_1, \ldots, e_m\} \) be the set of \( G \)'s edges and let \( \{1, \ldots, n\} \) be the set of \( G \)'s vertices. As before, we assume that both \( n \) and \( m \) are nonzero and that \( k \leq \min(n, m) \). Using Lemma 4.8, we can build an election \( E' = (C, V') \) with the following properties:

1. \(|C| = 2\ell^2 + \ell\), where \( \ell \geq 2n + 2m \) and \( \ell \) is polynomially bounded in \( \max(n, m) \);
2. \( \{p, r, e_1, \ldots, e_m\} \subseteq C \) (the remaining candidates are used for padding);
3. \( \text{score}^\alpha_{E'}(p) = 2\ell^2 - 1 \);
4. \( \text{score}^\alpha_{E'}(r) = 2\ell^2 - 1 - k + k\alpha \) in the nonunique-winner case \( (\text{score}^\alpha_{E'}(r) = 2\ell^2 - 1 - k + k\alpha - \epsilon \) in the unique-winner case\(^{12} \));
5. for each \( e_i \in C \), \( \text{score}^\alpha_{E'}(e_i) = 2\ell^2 - 1 + \alpha \) in the nonunique-winner case \( (\text{score}^\alpha_{E'}(e_i) = 2\ell^2 - 1 \) in the unique-winner case\);

\(^{12}\)Note that via the second paragraph of the proof it is easy to build an election where \( r \) has a score of this form.
6. the scores of all candidates in $C$ other than $p, r, e_1, \ldots, e_m$ are at most $2\ell^2 - n - 2$.

We form election $E = (C \cup D, V)$ via combining $E'$ with candidates $D = \{1, \ldots, n\}$ and appropriate voters such that the results of the head-to-head contests are:

1. $p$ ties with all candidates in $D$;
2. for each $e_j$, if $e_j$ is incident with some $i \in D$ then candidate $i$ defeats candidate $e_j$, and otherwise they tie;
3. all other candidates in $C$ defeat each of the candidates in $D$.

We will now show that $G$ contains a vertex cover of size at most $k$ if and only if there is a set $D' \subseteq D$ such that $p$ is a winner (the unique winner) of Copeland$^\alpha$ election $(C \cup D', V)$. It is easy to see that if $D'$ corresponds to a vertex cover of size at most $k$ then $p$ is a winner (the unique winner) of Copeland$^\alpha$ election $(C \cup D', V)$. The reason is that adding each member of $D'$ increases $p$'s score by $\alpha$, increases $r$'s score by one, and for each $e_j$, adding $i \in D'$ increases $e_j$'s score by $\alpha$ if and only if $e_j$ is not incident with $i$. Thus, via a simple calculation of the scores of the candidates, it is easy to see that $p$ is a winner (the unique winner) of this election.

On the other hand, assume that $p$ can become a winner (the unique winner) of Copeland$^\alpha$ election $(C \cup D', V)$ via adding some subset $D'$ of candidates from $D$. First, note that $\|D'\| \leq k$, since otherwise $r$ would end up with more points than (at least as many points as) $p$ and so $p$ would not be a winner (would not be the unique winner). We claim that $D'$ corresponds to a vertex cover of $G$. For the sake of contradiction, assume that there is some edge $e_j$ incident to vertices $u$ and $v$ such that neither $u$ nor $v$ is in $D'$. However, if this was the case then candidate $e_j$ would have more points than (at least as many points as) $p$ and so $p$ would not be a winner (would not be the unique winner). Thus, $D'$ must form a vertex cover of size at most $k$. $\blacksquare$

Note that in the above proof it is crucial that $\alpha$ is neither 0 nor 1. If $\alpha$ were 0 then the proof would fall apart because we would not be able to enforce that $D'$ is a vertex cover, and if $\alpha$ were 1 then we would not be able to limit the size of $D'$. In fact, we will now show in Theorem 4.11 that both Copeland$^0$ and Copeland$^1$ are vulnerable to control via adding an unlimited number of candidates (CCAC$_u$).

**Theorem 4.11** Let $\alpha \in \{0, 1\}$. Copeland$^\alpha$ is vulnerable to constructive control via adding an unlimited number of candidates (CCAC$_u$), in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.

**Proof.** Our input is candidate set $C$, spoiler candidate set $D$, a collection of voters with preferences (either preference lists or preference tables) over $C \cup D$, and a candidate $p \in C$. Our goal is to check whether there is some subset $D' \subseteq D$ such that $p$ is a winner (the
unique winner) of \((C \cup D', V)\) within Copeland\(^\alpha\). We will show that we can find such a set \(D'\), if it exists, by the following simple algorithm.

Let \(D_1 = \{d \in D \mid \text{score}_{(p,d),V}^\alpha(p) = 1\}\). Initialize \(D'\) to be \(D_1\), and delete every \(d \in D'\) for which \(\text{score}_{(C \cup D', V)}^\alpha(p) < \text{score}_{(C \cup D', V)}^\alpha(d)\). For the unique-winner problem, delete every \(d \in D'\) for which \(\text{score}_{(C \cup D', V)}^\alpha(p) \leq \text{score}_{(C \cup D', V)}^\alpha(d)\).

Clearly, this algorithm runs in polynomial time. To show that the algorithm works, first note that for all \(\hat{D} \subseteq D\), if \(p\) is a winner (the unique winner) of \((C \cup \hat{D}, V)\), then \(p\) is a winner (the unique winner) of \((C \cup (\hat{D} \cup D_1), V)\). This is so, because by Observation 4.2,

\[
\text{score}_{(C \cup \hat{D}, V)}^\alpha(p) = \text{score}_{(C \cup (\hat{D} \cap D_1), V)}^\alpha(p) + \sum_{d \in \hat{D} - D_1} \text{score}_{(p,d),V}^\alpha(p)
\]

Now suppose that for some \(\hat{D} \subseteq D_1\), \(p\) is a winner (the unique winner) of \((C \cup \hat{D}, V)\), but that the algorithm computes a set \(D'\) such that \(p\) is not a winner (not a unique winner) of \((C \cup D', V)\). We first consider the case that \(\hat{D} \subseteq D'\). Since \(p\) is not a winner (not a unique winner) of \((C \cup D', V)\), it follows by the construction of \(D'\) that there exists a candidate \(d \in C - \{p\}\) such that \(\text{score}_{(C \cup D', V)}^\alpha(p) < \text{score}_{(C \cup D', V)}^\alpha(d)\) \((\text{score}_{(C \cup D', V)}^\alpha(p) \leq \text{score}_{(C \cup D', V)}^\alpha(d))\). However, then in the nonunique-winner model we have

\[
\text{score}_{(C \cup D', V)}^\alpha(p) = \text{score}_{(C \cup \hat{D}, V)}^\alpha(p) + \|D'\| - \|\hat{D}\|
\]

\[
\geq \text{score}_{(C \cup \hat{D}, V)}^\alpha(d) + \|D'\| - \|\hat{D}\| \geq \text{score}_{(C \cup D', V)}^\alpha(d),
\]

which is a contradiction. In the unique-winner model in the above inequality the first \(\geq\) becomes a \(>\) and we reach a contradiction as well.

Finally, consider the case that \(\hat{D} \not\subseteq D'\). Let \(d\) be the first candidate in \(\hat{D}\) that is deleted from \(D'\) in the algorithm. Then there is a set \(D''\) such that \(\hat{D} \subseteq D'' \subseteq D_1\) and \(\text{score}_{(C \cup D'', V)}^\alpha(p) < \text{score}_{(C \cup D'', V)}^\alpha(d)\) \((\text{score}_{(C \cup D'', V)}^\alpha(p) \leq \text{score}_{(C \cup D'', V)}^\alpha(d))\). Since \(\hat{D} \subseteq D'' \subseteq D_1\), \(\text{score}_{(C \cup \hat{D}, V)}^\alpha(p) = \text{score}_{(C \cup D'', V)}^\alpha(p) - (\|D''\| - \|\hat{D}\|) \leq \text{score}_{(C \cup \hat{D}, V)}^\alpha(d)\). It follows that \(p\) is not a winner (the unique winner) of \((C \cup \hat{D}, V)\). This is again a contradiction.

The remainder of this section is dedicated to showing that for any rational \(\alpha\) such that \(0 \leq \alpha \leq 1\), Copeland\(^\alpha\) is resistant to constructive control both via deleting candidates and via partitioning candidates (with or without run-off and in both the TE and the TP model). We first handle the case of constructive control via deleting candidates (CCDC) and then, using our proof for the CCDC case as a building block, we handle the constructive partition-of-candidates cases. It would be best if the proofs for these partition cases used the CCDC construction as a black box but we have not found a way to do so. Thus, when handling
the partition cases we do use some of the specific properties of our CCDC construction but we do keep such dependencies to minimum.

**Theorem 4.12** Let $\alpha$ be a rational number such that $0 \leq \alpha \leq 1$. Copeland$^\alpha$ is resistant to constructive control via deleting candidates (CCDC), in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.

**Proof.** The proof follows via a reduction from the vertex cover problem. We first handle the nonunique-winner case.

Let $(G, k)$ be a given input instance of the vertex cover problem, where $G$ is an undirected graph and $k$ is the upper bound on the size of the vertex cover that we seek. Let $V(G) = \{1, \ldots, n\}$ and let $E(G) = \{e_1, \ldots, e_m\}$. As usual, we assume that $n$ and $m$ are nonzero and that $k < \min(n, m)$. We build election $E' = (C', V')$, where $C' = \{p, r, e_1, \ldots, e_m, 1, \ldots, n\}$ and the voter set $V'$ yields the following results of head-to-head contests (see Lemma 4.6):

1. $p$ defeats $r$,
2. $r$ defeats each candidate $e_i \in C'$,
3. each candidate $e_i \in C$ defeats exactly those two candidates $u, v \in \{1, \ldots, n\}$ that the edge $e_i$ is incident with,
4. each candidate $u \in \{1, \ldots, n\}$ defeats $p$ and all candidates $e_i \in C'$ such that vertex $u$ is not incident to $e_i$,
5. all the remaining contests result in a tie.

Let $\ell = n + m$. We form an election $E = (C, V)$ via combining election $E'$ with election Pad$^\ell = (C'', V'')$, where $C'' = \{t_0, \ldots, t_{2\ell}\}$ and the set $V''$ of voters is set as in Lemma 4.7. We select the following results of head-to-head contests between the candidates in $C'$ and the candidates in $C''$: $p$ and all candidates $e_i \in C'$ defeat everyone in $C''$ and each candidate in $C''$ defeats all candidates in $C' - \{p, e_1, \ldots, e_m\}$. It is easy to verify that election $E$ yields the following Copeland$^\alpha$ scores:

1. $\text{score}_E^\alpha(p) = m\alpha + 1 + 2\ell + 1$;
2. $\text{score}_E^\alpha(r) = m + n\alpha$;
3. for each $e_i \in C$, $\text{score}_E^\alpha(e_i) = m\alpha + 2 + 2\ell + 1$;
4. for each $i \in C$, $\text{score}_E^\alpha(i) \leq 1 + m + n\alpha$;
5. for each $t_i \in C$, $\text{score}_E^\alpha(t_i) = \ell + n + 1$.

The set of winners of $E$ is $W = \{e_1, \ldots, e_m\}$. We claim that $p$ can become a winner of Copeland$^\alpha$ election $E$ via deleting at most $k$ candidates if and only if the graph $G$ has a vertex cover of size at most $k$. In fact, $p$ is the only nonwinner of $E$ that can become a
winner after deleting up to \( k \) candidates. All the other ones lose by more than \( n \) points to the members of \( W \) and \( k < n \).

We now show that if \( p \) can become a winner via deleting at most \( k \) candidates then there is a set \( D \subseteq \{1, \ldots, n\} \) such that \( \|D\| \leq k \) and \( p \) is a winner of election \((C - D, V)\). Let \( D' \) be a smallest subset of \( C \) (cardinality-wise) such that \( p \) is a winner of election \((C - D', V)\). Clearly, no candidate in \( C'' \) belongs to \( D' \) because each member \( C'' \) loses his or her head-to-head contests with each member of \( \{r\} \cup W \) so deleting any such candidate from \( C \) does not bring \( p \) any closer to being a winner. Similarly, \( r \) wins all head-to-head contests with the members of \( W \) and so \( D' \) does not contain \( r \). Thus, \( D' \subseteq \{1, \ldots, n, e_1, \ldots, e_m\} \). Let us assume that there is some \( e_i \in D' \) such that in \( e_i \) is an edge incident to vertices \( u \) and \( v \) in \( G \). Via simple calculation we see that \( D' \) does not contain either \( u \) or \( v \) because then \( p \) would be a winner of election \((C - (D' - \{e_i\}), V)\), contradicting that \( D' \) is a smallest set with this property. However, then \( p \) is a winner of election \((C - ((D' \cup \{u\}) - \{e_i\})) \). Thus, via removing all members of \( \{e_1, \ldots, e_m\} \) from \( D' \) and replacing each of them with one of the nodes they are incident with we can build a set \( D \subseteq \{1, \ldots, n\} \) such that \( \|D\| \leq k \) and \( p \) is a winner of election \((C - D, V)\).

We will now argue that the set \( D \) from the previous paragraph corresponds to a vertex cover of \( G \). In election \( E \) each of \( e_1, \ldots, e_m \) has exactly one Copeland\(^\alpha \) point of advantage over \( p \). Deleting any candidate \( u \) corresponding to a vertex of \( G \) does not affect \( p \)'s score but it does decrease by one the scores of all the candidates \( e_1, \ldots, e_m \) that correspond to the edges incident with \( u \). Since deleting the candidates in \( D \) makes \( p \) a winner and since \( D \) contains only up to \( k \) candidates that correspond to vertices of \( G \), it must be the case that \( D \) corresponds to a vertex cover of \( G \). On the other hand, it is easy to see that if \( G \) has a vertex cover of size at most \( k \) then deleting the candidates that correspond to this vertex cover guarantees \( p \)'s victory. Thus, our reduction is correct and, as easily seen, computable in polynomial time. The proof for the nonunique-winner case is complete.

For the proof in the unique-winner case, we need to add one more candidate, \( \hat{r} \), that is a “clone” of \( r \) (i.e., \( \hat{r} \) ties in the head-to-head contest with \( r \) and has the same results as \( r \) in all other head-to-head contests). In such a modified election, \( p \) has the same Copeland\(^\alpha \) score as each of the \( e_i \)'s and has to gain at least one point over each of them to become the unique winner. The rest of the argument remains the same. \( \square \)

We will use the above construction in the resistance proofs for the cases of control via partition of candidates (with or without run-off, in TP and TE) cases proofs below. In particular, we will need the fact that in this construction the only candidates that may be winners after deleting at most \( k \) candidates are members of the set \( \{p\} \cup W \) (recall that \( W = \{e_1, \ldots, e_m\} \)).

**Theorem 4.13** Let \( \alpha \) be a rational number such that \( 0 \leq \alpha \leq 1 \). Copeland\(^\alpha \) is resistant to constructive control via run-off partition of candidates in both the ties-promote model (CCRPC-TP) and the ties-eliminate model (CCRPC-TE), in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.
Proof. The proof follows via a series of reductions from the vertex cover problem to appropriate variants of the CCRPC problem for Copeland\(^\alpha\) (i.e., to CCRPC-TP and CCRPC-TE in both the nonunique-winner model and in the unique-winner model).

Let \((G, k)\) be an instance of the vertex cover problem, where \(G\) is an undirected graph and \(k\) is the upper bound on the size of the vertex cover that we seek. As before, we let \(V(G) = \{1, \ldots, n\}\) and \(E(G) = \{e_1, \ldots, e_m\}\). Our goal is to build an election \(E\) in which our favorite candidate can become a winner (the unique winner) via run-off partitioning of candidates (with either the TP or the TE model) if and only if \(G\) has a vertex cover of size at most \(k\), and we do so via combining elections \(F\) and \(H\) as in Construction 4.14 below. We will later specify \(F\) and \(H\) separately for each of the two variants of the Copeland\(^\alpha\) CCRPC problem (TP and TE), but before doing so we will outline our proof in more detail and prove a useful property of Construction 4.14 (see Lemma 4.15).

Construction 4.14 Let \(F\) and \(H\) be two elections, \(F\) with candidates \(f_1 = p, f_2, \ldots, f_n\) and \(H\) with candidates \(r, h_1, \ldots, h_q\), \(q \geq 2\). We form election \(E = (C, V)\), where \(C = \{r, f_1, \ldots, f_n, h_1, \ldots, h_q\}\), via combining \(F\) and \(H\) and setting the results of head-to-head contests between candidates of \(F\) and \(H\) as follows:

1. For each \(f_i \in C\), \(f_i\) defeats \(r\).
2. For each \(h_i, f_j \in C\), \(h_i\) defeats \(f_j\).

In the next lemma we will show that the only partitions \((C_1, C_2)\) of \(C\) such that \(p\) is a winner (the unique winner) of the resulting Copeland\(^\alpha\) run-off election are of the form \(C_1 = F - D\), \(C_2 = H \cup D\), where \(D\) is a subset of \(F - \{p\}\), \(^{13}\) Next we will specify two variants of the election \(H\), one for the TP case and one for the TE case, such that the only partitions of the form presented above that may possibly lead to \(p\) being a winner (the unique winner) have \(\|D\| \leq k\). We will conclude the construction with selecting \(F\) to be one of the elections from the proof of Theorem 4.12, so that \(p\) can become a winner (the unique winner) of \(F\) via deleting at most \(k\) candidates if and only if \(G\) has a vertex cover of size at most \(k\).

Let \(E, F,\) and \(H\) be elections as in Construction 4.14. We further assume that there are no ties among the candidates of \(H\) (and that thus there are no ties among the corresponding candidates in \(E\)) and that in the case of CCRPC-TE, \(H - \{r\}\) has a unique winner. Let us assume that \(p\) can become a winner (the unique winner) of election \(E\) via a run-off partition of candidates and let \((C_1, C_2)\) be a partition of candidates such that \(p \in C_1\) is a winner (the unique winner) of a thus-formed Copeland\(^\alpha\) run-off election (performed using either the TP or the TE tie-handling rule).

Lemma 4.15 Both for the TP model and for the TE model it holds that there is a set \(D \subseteq F - \{p\}\) such that \(C_1 = F - D\) and \(C_2 = H \cup D\).

\(^{13}\)Recall that in this section we often identify elections with their candidate sets. Also, clearly, we can reverse the roles of \(C_1\) and \(C_2\) but for the sake of simplicity and without the loss of generality we will assume that \(p\) is in \(C_1\).
Proof. We will handle the TP and TE cases in parallel. For the sake of a contradiction, let us assume that $C_1 \cap H \neq \emptyset$. We consider three cases.

**Case 1:** $C_1$ contains at least two candidates from $H - \{r\}$. Let $h_i$ and $h_j$ be two such candidates such that $h_i$ wins his or her head-to-head contest with $h_j$. Note that $p$ is not a winner of $(C_1, V)$ because $score^\alpha_{(C_1, V)}(h_i) \geq \|C_1 \cap F\| + 1$ whereas $score^\alpha_{(C_1, V)}(p) \leq \|C_1 \cap F\|$ (since $p$ gets $\|C_1 \cap F\| - 1$ points from defeating the members of $\|C_1 \cap F\|$ and possibly one additional point from defeating $r$ if $r \in C_1$). Thus, in this case, $p$ does not advance to the final stage, irrespective of the tie-handling model used.

**Case 2:** $C_1$ contains exactly one member of $H - \{r\}$, say $h_i$. Via an analysis similar to the one above, if $C_1$ does not contain $r$ then $score^\alpha_{(C_1, V)}(p) < Copeland^\alpha_{(C_1, V)}(h_i)$ and $p$ does not advance to the final stage irrespective of the tie-handling model. If $r \in C_1$ then $score^\alpha_{(C_1, V)}(p) \leq Copeland^\alpha_{(C_1, V)}(h_i)$. Thus in the TE model $p$ certainly does not advance to the final stage. In the TP model the set of candidates that advance from $(C_1, V)$ to the final stage includes $h_i$ and it might include $p$. However, the final stage includes at least one more member of $H - \{r\}$ except $h_i$, namely a winner of $(C_2, V)$ (it is easy to verify that in our current case $(C_2, V)$ has at least one winner that belongs to $H - \{r\}$). Thus, either $p$ does not participate in the final stage or, via the same argument as in Case 1, $p$ is not a winner of the final stage because he or she meets there at least two members of $H - \{r\}$.

**Case 3:** $C_1 \cap H = \{r\}$. Since $r$ loses the head-to-head contests with all members of $F$, $r$ certainly does not advance to the final stage of the election. Let us assume that $p$ participates in the final stage. However, at least one winner (the only winner, in the TE model\(^\text{14}\)) of $(C_2, V)$ is a member of $H - \{r\}$. Then, via the same argument as in the first subcase of Case 2 above we can see that $p$ is not a winner of the final stage.

Thus the lemma holds.  

\(\blacksquare\)  
Lemma 4.15

We now define variants of election $H$ appropriate for the TE and TP models, in the nonunique-winner model, such that the set $D$ in Lemma 4.15 is forced to have at most $k$ elements. (We will handle the unique-winner cases at the end of the proof.)

For the TP case, we set $H'$ to be an election with candidate set \{\(r, h_1, \ldots, h_q\)\}, $q \geq 3$, such that there exists a nonnegative integer $\ell$ such that we have the following scores:

1. $score^\alpha_H(r) = \ell$;
2. $score^\alpha_H(h_1) = \ell - k - 1$;
3. $score^\alpha_H(h_2) = \ell - k - 1$;
4. for each $i \in \{3, \ldots, q\}$, $score^\alpha_H(h_i) \leq \ell - k - 1$.

\(^\text{14}\)Recall that we assumed that in the TE case $H - \{r\}$ has a unique winner and that each member of $H$ defeats each member of $F$ in their head-to-head contests.
Such an election is easy to build in polynomial time using Lemma 4.8.

For the TE case, we set $H''$ to have candidate set $\{r, h_1, \ldots, h_q\}$, $q \geq 2$ with the following scores:

1. $\text{score}_H^\alpha(r) = \ell$;
2. $\text{score}_H^\alpha(h_1) = \ell - k$;
3. for each $i \in \{2, \ldots, q\}$, $\text{score}_H^\alpha(h_i) < \ell - k - 1$.

Note that both $H'$ and $H''$ satisfy the assumptions that we made regarding $H$ before Lemma 4.15.

**Lemma 4.16** For the TP case with $H = H'$ and for the TE case with $H = H''$, the set $D$ in Lemma 4.15 has the additional property that $\|D\| \leq k$.

**Proof.** Recall that in $E$ each candidate $h_i \in H - \{r\}$ wins each of his or her head-to-head contests with candidates in $F$ and that each candidate $f_i \in F$ wins his or her head-to-head contest with $r$. From Lemma 4.15 we know that $C_1 = F - D$ and $C_2 = H \cup D$. If $\|D\| > k$ in the TP case, then $(C_2, V)$ has two winners, $h_1$ and $h_2$ and even if $p$ were promoted to the final stage, he or she would meet two members of $H - \{r\}$ there and would not become a global winner. If $\|D\| > k$ in the TE case, then $h_1$ would be the unique winner of $(C_2, V)$ and the final stage would involve, at best, $p$ and $h_1$. Naturally, $p$ would lose. \[\Box\] Lemma 4.16

For the TP case (TE case), we set $F$ to be the election built in the proof of Theorem 4.12 for the nonunique-winner model (for the unique-winner model); we keep the candidate names as in that proof, i.e., in particular candidates $e_1, \ldots, e_m$ correspond to the edges of graph $G$. Since $p$ can become a winner of his or her subcommittee in the TP model (in the TE model) if and only if $p$ can become a winner (the unique winner) of election $F - D$, where $D \subseteq F - \{p\}$ and $\|D\| \leq k$, it follows by our choice of $F$, that $p$ can advance to the final stage only if $G$ has a vertex cover of size at most $k$. On the other hand, if $G$ has a vertex cover of size at most $k$ then it is easy to see that if we partition the candidates in $C$ as in Lemma 4.15 with $D$ containing the candidates corresponding to an at-most-size-$k$ vertex cover of $G$ then $p$ advances to the final stage of the election and is a winner there. This is because the following hold. In the TP case the subcommittee $H \cup D$ has $r$ as the unique winner and the subcommittee $F - D$‘s winner set contains $p$ and some subset of $\{e_1, \ldots, e_m\}$ (see the note below the proof of Theorem 4.12). Since $p$ and all candidates in $\{e_1, \ldots, e_m\}$ tie in their head-to-head contests and since they all defeat $r$, they all are winners of the final stage. Similarly, in the TE case, the subcommittee $H \cup D$ either has no winner or has $r$ as the unique winner and the subcommittee $F - D$ has $p$ as the unique winner. Since $p$ defeats $r$, $p$ is the winner of the final stage. This completes the proof for the nonunique-winner case.

The proof for the TE case in the nonunique-winner model actually works just as well in the unique-winner model and so it remains to handle the TP case in the nonunique-winner
model. To do so, we form the election $E$ using Construction 4.14 with $F$ set to the unique-winner version of the election from Theorem 4.12 and with $H$ set to $H'$. Via Lemmas 4.15 and 4.16 and the subsequent discussion we have that any partition of $E$’s candidate set $C$ into $C_1$ and $C_2$, where $p \in C_1$ is a winner of the final-stage election requires $C_1 = F - D$ and $C_2 = H \cup D$, where $D$ corresponds to an at-most-size-$k$ vertex cover of $G$. On the other hand, by the choice of $F$, it is easy to see that using such a $D$ guarantees that $p$ is the unique winner of the final stage election. The proof is complete. \[\square\]  

Theorem 4.13

Copeland$^\alpha$ is also resistant to constructive control via partition of candidates (without run-off) for each rational value of $\alpha$ between (and including) 0 and 1. However the proofs for the TP and TE cases are not as uniform as in the CCRPC case and so we treat these cases separately.

Theorem 4.17

Let $\alpha$ be a rational number such that $0 \leq \alpha \leq 1$. Copeland$^\alpha$ is resistant to constructive control via partition of candidates with the ties-promote tie-handling rule (CCPC-TP), in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.

**Proof.** The proof is very similar to that of Theorem 4.13 and we maintain similar notation. In particular, $(G, k)$ is the given instance of the vertex cover problem we reduce from (where $V(G) = \{1, \ldots, n\}$ and $E(G) = \{e_1, \ldots, e_m\}$), and we set $F$ to be the nonunique-winner variant of the election built in the proof of Theorem 4.12 when we handle the nonunique-winner case for Copeland$^\alpha$-CCPC-TP (we set $F$ to be the unique-winner variant of that election when we handle the unique-winner case for Copeland$^\alpha$-CCPC-TP).

We define $H$ as in the proof of Theorem 4.13 for the TP case, except for the following changes. Now, candidates $h_1$, $h_2$, and $h_3$ all have score $\ell - k - 1$ (candidates $h_4, \ldots, h_q$ receive at most $\ell - k - 1$ points) and the results of the head-to-head contests between $r, h_1, h_2,$ and $h_3$ are that $h_1$ defeats $h_2, h_2$ defeats $h_3$, $h_3$ defeats $h_1$, and $r$ defeats each of $h_1, h_2,$ and $h_3$.\[\text{Note that } r \text{ has score } \ell \text{ as before. Such an election } H \text{ is easy to build using Lemma 4.8.}\]

Election $E = (C, V)$ is formed via applying Construction 4.14 to elections $F$ and $H$.

In constructive control via partition of candidates (CCPC) the first stage of the election is held among some subset $C' \subseteq C$ of candidates. Then the winners of the first stage (TP model) compete with the candidates in $C - C'$. The following two lemmas describe what properties $C'$ needs to satisfy if $p$ is to be a winner (the unique winner) of the final stage.

**Lemma 4.18** Let $C'$ be a subcommittee such that $p$ is a winner (the unique winner) of the final stage in $E$ with subcommittee $C'$. It holds that $p$ is not a member of $C'$.

**Proof.** If $p$ was in $C'$ together with at least two members of $H$ then $p$ would not be a winner of this subelection. If $p$ were in $C'$ with less than two members of $H$ then $p$ would

\[\text{Note that for such a cycle of head-to-head contest results is that if we delete at most one candidate from } H \text{ then there still will be at least one candidate with score } \ell - k - 1.\]
either meet at least two members of $H$ in the final stage, or $p$ would not make it to the final stage. In either case, $p$ would not be a winner of the final stage. \(\square\) Lemma 4.18

**Lemma 4.19** Let $C'$ be a subcommittee such that $p$ is a winner (the unique winner) of $E$ with subcommittee $C'$. It holds that $H \subseteq C'$.

**Proof.** From Lemma 4.18 we know that $p$ is not in $C'$. If more than two members of $H$ were not in $C'$ then $p$ would meet at least two members of $h$ in the final stage and so would not be a winner.

Assume that exactly one member of $H$, say $h$, is not in $C'$. If $r$ is not in $C'$ then the set of winners of $(C', V)$ includes at least one member of $H - \{r\}$ and so $p$ meets two members of $H - \{r\}$ in the final stage and thus is not a winner of that stage. So we additionally assume that $r$ is in $C'$. Since $H - \{h\} \subseteq C'$, at least two of $h_1$, $h_2$, and $h_3$ are in $C'$. If $C'$ contains more than $k$ candidates from $F$ then at least one of $h_1$, $h_2$, and $h_3$ is a winner of $(C', V)$ and $p$, again, competes (and loses) against two members of $H$ in the final election. Thus, let us assume that $C'$ contains at most $k$ members of $F$, call them $d_1, d_2, \ldots, d_j$, where $j \leq k$. In such a case, $p$ loses to at least one candidate $s$ in $F - \{d_1, \ldots, d_j\}$ (recall that $k < n$ because otherwise the input vertex cover instance is trivial) and so in the final stage $p$ has a score lower than $h$'s score: $h$ wins the head-to-head contests with everyone with whom $p$ wins his or her head-to-head contests except for $r$, but $h$ also wins his or her head-to-head contests with both $p$ and $s$. As a result, $p$ does not win the final stage. This completes the proof. \(\square\) Lemma 4.19

As an immediate corollary we have that if $p$ is a winner of our two-stage election then the first stage involves candidates in a set $C'$ such that $C' = H \cup D$, where $D \subseteq F - \{p\}$ and $\|D\| \leq k$.

**Lemma 4.20** Let $C'$ be a subcommittee such that $p$ is a winner (the unique winner) of the final stage of $E$ with subcommittee $C'$. Then $C'$ is of the form $H \cup D$, where $D \subseteq F - \{p\}$ and $\|D\| \leq k$ and $p \notin D$.

Since $r$ is the unique winner of any subelection of the form $H \cup D$, where $D$ contains at most $k$ candidates from $F - \{p\}$ and all members of $F$ defeat $r$ in their head-to-head contests, via Lemma 4.20, we have that $p$ can become a winner (the unique winner) if and only if we can select a set $D$ of up to $k$ candidates from $F$ such that $p$ is a winner (the unique winner) of $F - D$. By the choice of $F$ and Theorem 4.12, this is possible only if $G$ has a vertex cover of size at most $k$. This completes the proof. \(\square\) Theorem 4.17

We now turn to the TE variant of constructive control via partitioning candidates. We start by showing resistance for Llull’s system (i.e., for Copeland\(^1\)).

**Theorem 4.21** Copeland\(^1\) is resistant to constructive control via partition of candidates with the ties-eliminate tie-handling rule (CCPC-TE), in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.
Proof. We use the same reduction as that in the proof of Theorem 4.17, except that we now use a slightly different variant of the election $H$.

Let $(G, k)$ be our input instance of the vertex cover problem. As in the proof of Theorem 4.17, we form an election $E$ via combining elections $F$ and $H$, where $F$ and $H$ are as follows. If we are in the nonunique-winner mode then $F$ is the nonunique-winner variant of the election from the proof Theorem 4.12, and otherwise it is the unique-winner variant of that election. $H$ is an election whose candidate set is $\{r, h_1, \ldots, h_\ell\}$ and whose voter set is such that $r$ ties all head-to-head contests with the candidates $h_1, \ldots, h_\ell$ and the scores of the candidates satisfy:

1. $\text{score}_H^1(r) = \ell$;
2. $\text{score}_H^1(h_1) = \ell - k$;
3. for each $h \in H - \{r, h_1\}$, $\text{score}_H^1(h) < \ell - k$.

It is easy to see that such an election can be built in polynomial time in $k$ (thus ensuring that $\ell$ is polynomially bounded in $\|k\|$), even though Lemma 4.8 cannot be directly used for this purpose.\footnote{Recall that, as before, we tacitly assume that $k < n$, where $n$ is the number of vertices in $G$.}

It is easy to see that if $G$ does have a vertex cover of size $k$ then using the subcommittee $H \cup D$, where $D$ is a subset of candidates from $F$ that corresponds to this vertex cover does make $p$ a winner (the unique winner) of the final stage of our two-stage election.

For the converse, we show that if $p$ can be made a winner (the unique winner) via partition of candidates then $G$ does have a vertex cover of size at most $k$. First, we note that $p$ is not a winner (and so certainly not a unique winner) of any subelection that involves any of the candidates $h_1, \ldots, h_\ell$. (This is because each of $h_1, \ldots, h_\ell$ wins all of his or her head-to-head contests with members of $F$ and ties the head-to-head contest with $r$.) Thus if $p$ is a winner (the unique winner) of our two-stage election then all candidates $h_1, \ldots, h_\ell$ participate in the first-stage subcommittee and $p$ does not. In this case the subcommittee also contains $r$ because otherwise $h_1$ would be the unique winner of the subcommittee and, via the previous argument, $p$ would not be a winner of the final stage. Similarly, if the subcommittee contained more than $k$ members of $F$ then, again, $h_1$ would be its unique winner preventing $p$ from being a winner of the final stage.

Let $C'$ be a subcommittee (i.e., the set of the candidates that participate in the first stage of the election) such that $p$ is a winner (the unique winner) of our two-stage election. Via the previous paragraph it holds that $C' = H \cup D$, where $D \subseteq F$ and $\|D\| \leq k$. Either $r$ is the unique winner of $C'$ (if $\|D\| < k$) or $C'$ does not have a unique winner. Thus the winner set of the final stage is the same as that of $F - D$ because all candidates in $F - D$ defeat $r$ in their head-to-head contests. As a result, via Theorem 4.12, we have that $p$ is a winner (the unique winner) if and only if $D$ corresponds to an up-to-size-$k$ vertex cover of $G$. \qed
Theorem 4.22 Let $\alpha$ be a rational number, $0 \leq \alpha < 1$. Copeland$^\alpha$ is resistant to constructive control via partition of candidates with the ties-eliminate tie-handling rule (CCPC-TE), in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.

Proof. The proof follows via a reduction from the vertex cover problem. Let $(G, k)$ be our input instance where $G$ is an undirected graph and $k$ is the upper bound on the size of the vertex cover that we seek. Via combining two subelections, $F$ and $H$, we will build an election $E$ such that a candidate $p$ in $E$ can become a winner (the unique winner) via partitioning of candidates if and only if $G$ has a vertex cover of size at most $k$.

In the nonunique-winner case we take $F$ to be the nonunique-winner variant of the election built in the proof of Theorem 4.12. In the unique-winner case we set $F$ to be the unique-winner variant. Election $H$ has candidate set $\{r, \hat{r}, h_1, \ldots, h_k\}$ and a voter set that yields the following results of head-to-head contests within $H$.

1. $r$ and $\hat{r}$ are tied;
2. $r$ ties with every candidate $h_i$, $i \in \{1, \ldots, k\}$;
3. $\hat{r}$ defeats every candidate $h_i$, $i \in \{1, \ldots, k\}$;
4. The results of head-to-head contests between candidates $h_1, \ldots, h_k$ are set arbitrarily.

Within election $H$ the candidates have the following Copeland$^\alpha$ scores:

1. $\text{score}_H^\alpha(r) = k\alpha + \alpha$;
2. $\text{score}_H^\alpha(\hat{r}) = k + \alpha$;
3. for each $i \in \{1, \ldots, k\}$, $\text{score}_H^\alpha(h_i) \leq k - 1 + \alpha$.

We form election $E = (C, V)$ via combining elections $F$ and $H$ in such a way that $r$ defeats all the candidates in $F$ and for all the other head-to-head contests between the candidates from $F$ and the candidates from $H$ the result is a tie.

It is easy to see that if $G$ does have a vertex cover of size at most $k$ then we can make $p$ a winner (the unique winner) via partitioning of candidates. To do so we hold the first-stage election among the candidates in $H \cup D$, where $D$ is a set of candidates in $F$ that correspond to a vertex cover within $G$ of size at most $k$. This subcommittee either has $\hat{r}$ as a unique winner (if $D$ contains less than $k$ candidates) or does not have a unique winner (if $\|D\|$ is exactly $k$). In either case, via our choice of $F$ and the fact that all members of $F$ tie with $\hat{r}$, it holds that $p$ is a winner (the unique winner) of the final stage.

For the converse, let us assume that $p$ can become a winner (the unique winner) via partitioning of candidates and let $C'$ be a subset of candidates such that if the first-stage election is $(C', V)$ then $p$ is a winner (the unique winner) of the final stage. If $p$ and $r$ participate in the same subelection (be it the first stage or the final stage) then it is easy to see that $p$ is not a winner of that subelection. (This is true because $r$ wins his or her
head-to-head contests with all members of $F$ and ties with everyone else, whereas $p$ ties with all candidates $\hat{r}, h_1, \ldots, h_k$ but, naturally, loses to $r$.) Thus $r \in C'$ and $p \notin C'$.

We will now show that $C'$ contains at most $k$ members of $F$. For the sake of contradiction let us assume that this is not the case and let $d$ be the number of members of $F$ in $C'$, $d > k$, and let $k'$ be the number of members of $H$ in $C'$. Let us rename the candidates in $F$ such that $f_1, \ldots, f_d$ are those that belong to $C'$.

Finally, we set $b = 0$ if $\hat{r} \notin C'$, and $b = 1$ otherwise. We have the following Copeland$^\alpha$ scores within $C'$:

1. $\text{score}_{C'}^\alpha(r) = d + k' \alpha + b \alpha$;
2. $\text{score}_{C'}^\alpha(\hat{r}) = k' + d \alpha + \alpha$, provided that $\hat{r} \in C'$;
3. for each $h_i \in C'$, $\text{score}_{C'}^\alpha(h_i) \leq k' - 1 + d \alpha + \alpha$;
4. for each $f_i \in C'$, $\text{score}_{C'}^\alpha(f_i) \leq d - 1 + k' \alpha + b \alpha$.

Since $k' \leq k$ and $d > k$, it is easy to see that irrespective of whether or not $\hat{r}$ participates in $C'$, $r$ is the unique winner within his or her subcommittee and, as a result, $p$ is not a winner of the final stage. Thus $C'$ contains at most $k$ members of $F$.

Let $D$ be the set of candidates from $F$ that are in $C'$. Since $p$ is a winner (the unique winner) of the final stage, $r$ does not participate in that stage. Thus, the final stage is held with candidate set $(F - D) \cup H'$, where $H'$ is some subset of $H - \{r\}$. It is easy to see that the winners of $F - D$ have such high scores in addition to tying their head-to-head contests with all members of $H'$ that the winner set of $((F - D) \cup H', V)$ is the same as that of $(F - D, V)$. Thus, via Theorem 4.12, if $p$ is a winner (the unique winner) of $F - D$, $\|D\| \leq k$, then $D$ corresponds to a vertex cover of size at most $k$ in $G$. This completes the proof. \[\square\]

4.2 Voter Control

In this subsection we show that for each rational $\alpha$, $0 \leq \alpha \leq 1$, Copeland$^\alpha$ is resistant to all types of voter control. Table 2 lists for each type of voter control, each rational $\alpha$, $0 \leq \alpha \leq 1$, and each winner model (i.e., nonunique-winner model and unique-winner model) the theorem in which each given case is handled. We start with control via adding voters.

**Theorem 4.23** Let $\alpha$ be a rational number such that $0 \leq \alpha \leq 1$. Copeland$^\alpha$ is resistant to both constructive and destructive control via adding voters (CCA$^V$ and DCA$^V$), in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.

**Proof.** Our result follows via reductions from the X3C problem. We will first show how to handle the nonunique-winner constructive case and later we will argue that the construction can be easily modified for each of the remaining cases.
\[ \alpha = 0 \quad 0 < \alpha < 1 \quad \alpha = 1 \]

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Table 2: Table of theorems covering all resistance results for voter control for Copeland\(\alpha\). Each theorem covers both the case of rational voters and the case of irrational ones.

Let \((B, S)\) be an X3C instance where \(B = \{b_1, \ldots, b_{3k}\}\) and \(S = \{S_1, \ldots, S_n\}\) is a finite collection of three-element subsets of \(B\). The question is if one can pick \(k\) sets \(S_{a_1}, \ldots, S_{a_k}\) such that \(B = \bigcup_{i=1}^{k} S_{a_i}\).

We build a Copeland\(\alpha\) election \(E = (C, V)\) as follows. The candidate set \(C\) contains candidates \(p\) (the preferred candidate), \(r\) (\(p\)'s rival), \(s\), all members of \(B\), and some number of padding candidates. We select the voter set \(V\) so that we have the following Copeland\(\alpha\) scores for these candidates, where \(\ell\) is some large enough (but polynomially bounded in \(n\)) nonnegative integer:

1. \(\text{score}_E^\alpha(p) = \ell - 1\),
2. \(\text{score}_E^\alpha(r) = \ell + 3k\),
3. \(\text{score}_E^\alpha(s) < \ell - 1\), and
4. all other candidates have Copeland\(\alpha\) scores below \(\ell\).

Also, we have the following results of head-to-head contests between the candidates in \(C\):

1. \(\text{vs}_E(s, p) = k - 1\),
2. for each \(i \in \{1, \ldots, k\}\), \(\text{vs}_E(r, b_i) = k - 3\), and
3. for all other pairs of candidates \(c\), \(d\), we have \(|\text{vs}_E(c, d)| \geq k + 1\).

It is fairly easy to see that such an election can be built. \(^{17}\) One can obtain appropriate scores using Lemma 4.8 and then it is easy to add an appropriate number of voters to match the requirements regarding the relative scores. Since all the relative scores that we specify are of the same parity, this is possible.

\(^{17}\)Piotr, it is not at all clear to me how this election can be built; please add more detail. I also changed the name of candidate \(v\) to \(s\), since \(v\) is a bad name for a candidate. – EH

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We also specify a set $W$ of voters that the chair can possibly add. For each set $S_i \in S$ we have a single voter $w_i \in W$ with preference list:

$$p > B - S_i > r > S_i > \cdots$$

(all unmentioned candidates follow in any fixed arbitrary order). We claim that $(B, S)$ contains an exact cover if and only if $p$ can become a winner of the above election via adding at most $k$ voters selected from $W$.

If $(B, S)$ has a cover, say $S_{a_1}, \ldots, S_{a_k}$, then we can make $p$ a winner via adding the voters from $U = \{w_{a_1}, \ldots, w_{a_k}\}$. Adding these $k$ voters increases $p$'s score by one, since $p$ now beats $r$ in their head-to-head contest. Since voters in $U$ correspond to a cover, the score of $r$ goes down by $k$ points. Why is this so? For each $b_i \in B$, adding the $k - 1$ voters in $U$ that correspond to the sets in the cover not containing $b_i$ increases the relative performance of $b_i$ versus $r$ by $k - 1$ votes, thus giving $b_i$ two votes of advantage over $r$. Adding the remaining voter from $U$ decreases this advantage to 1, but still $b_i$ wins the head-to-head contest with $r$.

Let us show that if we can make $p$ a winner via adding at most $k$ voters then $(B, S)$ has a cover. Note that $p$ is the only candidate that can possibly become a winner via adding at most $k$ voters, that $p$ can at best obtain Copeland$^a$ score $\ell$, if we add exactly $k$ voters, and that $r$ can lose at most $3k$ points via losing his or her head-to-head contests with each of the $b_i$'s. Thus the only way for $p$ to become a winner via adding at most $k$ voters from $W$ is that we add exactly $k$ voters such that $r$ loses his or her head-to-head contest with each $b_i$. Assume that $U \subseteq W$ is such a set of voters that does not correspond to a cover of $B$. This means that there is some candidate $b_i$ such that at least two voters in $U$ prefer $r$ to $b_i$. However, if this is the case then $b_i$ cannot defeat $r$ in their head-to-head contest and $p$ is not a winner. Thus, we have that $U$ corresponds to a cover. This completes the proof of the nonunique-winner constructive variant of the theorem.

For the constructive unique-winner case we modify election $E$ so that $\text{score}_E(p) = \ell$. All other listed properties of the relative and absolute scores are unchanged. As in the previous case, it is easy to see that $p$ can become the unique winner via adding $k$ voters that correspond to a cover of $B$. For the converse, we will show that we still need to add exactly $k$ voters if $p$ is to become the unique winner.

If we added fewer than $k - 1$ voters then $p$ would not get any extra points and so it would be impossible for $p$ to become the unique winner. Let us now show that adding exactly $k - 1$ voters cannot make $p$ the unique winner. If we added exactly $k - 1$ voters then $p$ would get $\alpha$ points extra from the tie with $s$. Now consider some candidate $b_i \in S_j$, where $S_j$ corresponds to one of the added voters, $w_j$. Since $w_j$ prefers $r$ to $b_i$, adding $w_j$ to the election increases the relative performance of $r$ versus $b_i$ to $k - 2$. Thus adding the remaining $k - 2$ voters can result in $b_i$ either tieing or losing his or her head-to-head contest with $r$. In either case $p$ would not have a high enough score to become the unique winner. Thus, we know that exactly $k$ candidates must be added if we want $p$ to become the unique winner and, via the same argument as in the previous case, we know that they have to correspond to a cover.
For the destructive cases it suffices to note that the proof for the constructive nonunique-winner case works also as a proof for the destructive unique-winner case (where we are preventing \( r \) from being the unique winner) and the constructive unique-winner case works also as a proof for the destructive nonunique-winner case (where we are preventing \( r \) from being a winner).

Let us now turn to the case of control via deleting voters. Unfortunately, the proofs here are not as uniform as before and we need in some cases to handle \( \alpha = 1 \) separately from the case where \( 0 \leq \alpha < 1 \). Also, we cannot use the construction lemma (Lemma 4.8) anymore to so conveniently build our elections. In the case of deleting voters (or partitioning voters) we need to have a very clear understanding of how each voter affects the election and the whole point of introducing the construction lemma was to abstract from such low-level details.

Analogously to the case of candidate control, we will later re-use the resistance proofs for deleting voters within the resistance proofs for partitioning voters.

**Theorem 4.24** Copeland\(^1\) is resistant to constructive control via deleting voters (CCDV) in the nonunique-winner model and to destructive control via deleting voters (DCDV) in the unique-winner model, for each in both the rational and irrational voter model.

**Proof.** Let \((B, S)\) be an instance of X3C, where \( B = \{b_1, \ldots, b_{3k}\} \) and \( S = \{S_1, \ldots, S_n\} \) is a finite family of three-element subsets of \( B \). Without loss of generality, we assume that \( n \geq k \) and that \( k > 2 \) (if \( n < k \) then \( S \) does not contain a cover of \( B \), and if \( k \leq 2 \) then we can solve the problem by brute force). We build an election \( E = (C, V) \) such that the preferred candidate \( p \) can become a Copeland\(^1\) winner of \( E \) by deleting at most \( k \) voters if and only if \( S \) contains a \( k \)-element cover of \( B \).

We let the candidate set \( C \) be \( \{p, r, b_1, \ldots, b_{3k}\} \) and we let \( V \) be the following set of \( 4n - k \) voters:

1. We have \( n - 1 \) voters with preference \( B > p > r \);
2. we have \( n - k + 1 \) voters with preference \( p > r > B \);
3. for each \( S_i \in S \), we have two voters, \( v_i \) and \( v'_i \), such that
   - (a) \( v_i \) has preference \( r > B - S_i > p > S_i \);
   - (b) \( v'_i \) has preference \( r > S_i > p > B - S_i \).

It is easy to see that for all \( b_i \in B \), \( \text{vs}_E(r, b_i) = 2n - k + 2 \), \( \text{vs}_E(b_i, p) = k - 2 \), and \( \text{vs}_E(r, p) = k \).

If \( S \) contains a \( k \)-element cover of \( B \), say \( \{S_{a_1}, \ldots, S_{a_k}\} \), then we delete voters \( v_{a_1}, \ldots, v_{a_k} \). In the resulting election, \( p \) ties every other candidate in their head-to-head contests, and thus \( p \) is a winner.

For the converse, suppose that there is a subset \( W \) of at most \( k \) voters such that \( p \) is a winner of \( \tilde{E} = (C, V - W) \). It is easy to see that \( \text{score}^{1/2}_E(r) = 3k + 1 \). Since \( p \) is a winner of
\[ \hat{E}, p \] must tie-or-defeat every other candidate in their head-to-head contests. By deleting at most \( k \) voters, \( p \) can at best tie \( r \) in their head-to-head contest. And \( p \) will only tie \( r \) if \( \|W\| = k \) and every voter in \( W \) prefers \( r \) to \( p \). It follows that \( W \) is a size \( k \) subset of \( \{v_1, v'_1, \ldots, v_n, v'_n\} \).

Let \( b_i \in B \). Recall that \( \text{vs}_{\hat{E}}(b_i, p) = k - 2 \) and that \( p \) needs to at least tie \( b_i \) in their head-to-head contest in \( \hat{E} \). Since \( \|W\| = k \), it follows that \( W \) can contain at most one voter that prefers \( p \) to \( b_i \). Since \( k > 2 \), it follows that \( W \) contains only voters from the set \( \{v_1, \ldots, v_n\} \) and that the voters in \( W \) correspond to a \( k \)-element cover of \( B \).

This completes the proof for the nonunique-winner constructive case. This proof also handles the unique-winner destructive case, since \( r \) is always a winner after deleting at most \( k \) voters from \( E \) and \( b_i \) is never a winner after deleting at most \( k \) voters from \( E \). And so \( r \) can be made to not uniquely win by deleting at most \( k \) voters if and only if \( p \) can be made a winner by deleting at most \( k \) voters. \( \square \)

**Theorem 4.25** Let \( \alpha \) be a rational number such that \( 0 \leq \alpha \leq 1 \). Copeland\( ^{\alpha} \) is resistant to constructive control via deleting candidates (CCDV) in the unique-winner model and to destructive control via deleting candidates (DCDV) in the nonunique-winner model, for each in both the rational and irrational voter model.

**Proof.** As in the proof of the previous theorem, let \((B, S)\) be an instance of X3C, where \( B = \{b_1, \ldots, b_{3k}\} \) and \( S = \{S_1, \ldots, S_n\} \) is a finite family of three-element subsets of \( B \). Without loss of generality, we assume that \( n \geq k \) and that \( k > 2 \) (if \( n < k \), then \( S \) does not contain a cover of \( B \) and if \( k \leq 2 \), we can solve the problem by brute force). We build an election \( E = (C, V) \) such that

1. If \( S \) contains a \( k \)-element cover of \( B \), then the preferred candidate \( p \) can become the unique Copeland\( ^{\alpha} \) winner of \( E \) by deleting at most \( k \) voters, and
2. If \( r \) can become a non-winner by deleting at most \( k \) voters, then \( S \) contains a \( k \)-element cover of \( B \).

We use the election from the proof of Theorem 4.24 with one extra voter with preference \( p > r > B \). That is, we let the candidate set \( C \) be \( \{p, r, b_1, \ldots, b_{3k}\} \) and we let \( V \) be the following set of \( 4n - k + 1 \) voters:

1. We have \( n - 1 \) voters with preference \( B > p > r \);
2. We have \( n - k + 2 \) voters with preference \( p > r > B \);
3. For each \( S_i \in S \) we have two voters, \( v_i \) and \( v'_i \), such that
   (a) \( v_i \) has preference \( r > B - S_i > p > S_i \);
   (b) \( v'_i \) has preference \( r > S_i > p > B - S_i \).
It is easy to see that for all $b_i \in B$, $\text{vs}_E(r, b_i) = 2n - k + 3$, $\text{vs}_E(b_i, p) = k - 3$, and $\text{vs}_E(r, p) = k - 1$.

If $S$ contains a $k$-element cover of $B$, say $\{S_{a_1}, \ldots, S_{a_k}\}$, then we delete voters $v_{a_1}, \ldots, v_{a_k}$. In the resulting election, $p$ beats every other candidate in their head-to-head contests, and thus $p$ is the unique winner.

To prove the second statement, suppose that there is a subset $W$ of at most $k$ voters such that $r$ is not a winner of $\widehat{E} = (C, V - W)$. Since $\text{vs}_E(r, b_i) = 2n - k + 3$ and $n \geq k$, it is immediate that $r$ beats every $b_i \in B$ in their head-to-head contests in $\widehat{E}$. In order for $r$ not to be a winner of $\widehat{E}$, $p$ must certainly defeat $r$ and tie-or-defeat every $b_i \in B$ in their head-to-head contests. $p$ can only defeat $r$ in their head-to-head contest if $|W| = k$ and every voter in $W$ prefers $r$ to $p$. It follows that $W$ is a size $k$ subset of $\{v_1, v'_1, \ldots, v_n, v'_n\}$.

Let $b_i \in B$. Recall that $\text{vs}_E(b_i, p) = k - 3$ and that $p$ needs to at least tie $b_i$ in their head-to-head contest in $\widehat{E}$. Since $|W| = k$, it follows that $W$ can contain at most one voter that prefers $p$ to $b_i$. Since $k > 2$, it follows that $W$ contains only voters from the set $\{v_1, \ldots, v_n\}$ and that the voters in $W$ correspond to a $k$-element cover of $B$.

\[\Box\]

**Theorem 4.26** Let $\alpha$ be a rational number such that $0 \leq \alpha < 1$. Copeland$^\alpha$ is resistant to constructive control via deleting voters (CCDV) in the nonunique-winner model and to destructive control via deleting voters (DCDV) in the unique-winner model, for each in both the rational and irrational voter model.

**Proof.** Let $(B, S)$ be an instance of X3C, where $B = \{b_1, \ldots, b_{3k}\}$ and $S = \{S_1, \ldots, S_n\}$ is a finite family of three-element subsets of $B$. Without loss of generality, we assume that $n \geq k$ and that $k > 2$ (if $n < k$ then $S$ does not contain a cover of $B$, and if $k \leq 2$ then we can solve the problem by brute force). We build an election $E = (C, V)$ such that:

1. If $S$ contains a $k$-element cover of $B$, then the preferred candidate $p$ can become a Copeland$^\alpha$ winner of $E$ by deleting at most $k$ voters, and

2. if $r$ can be made to not uniquely win the election by deleting at most $k$ voters, then $S$ contains a $k$-element cover of $B$.

Our election is similar to the elections from Theorems 4.24 and 4.25. To avoid problems when $\alpha = 0$, we introduce a new candidate $\hat{r}$ to ensure that $p$ and $r$ are the only possible winners after deleting at most $k$ candidates. We let the candidate set $C$ be $\{p, r, \hat{r}, b_1, \ldots, b_{3k}\}$ and we let $V$ be the following set of $4n - k + 2$ voters:

1. We have $n - 2$ voters with preference $B > p > r > \hat{r}$; 
2. we have $n - k + 2$ voters with preference $p > r > \hat{r} > B$; 
3. for each $S_i \in S$ we have two voters, $v_i$ and $v'_i$, such that 
   (a) $v_i$ has preference $r > \hat{r} > B - S_i > p > S_i$;
(b) $v'_i$ has preference $r > \hat{r} > S_i > p > B - S_i$;

4. we have one voter with preference $r > p > \hat{r} > B$;

5. we have one voter with preference $B > p > r > \hat{r}$.

It is easy to see that for all $b_i \in B$, $vs_E(r, b_i) = vs_E(\hat{r}, b_i) = 2n - k + 4$, $vs_E(r, \hat{r}) = 4n - k + 2$, $vs_E(b_i, p) = k - 4$, $vs_E(r, p) = k$, and $vs_E(\hat{r}, p) = k - 2$.

If $S$ contains a $k$-element cover of $B$, say $\{S_{a1}, \ldots, S_{ak}\}$, then we delete voters $v_{a1}, \ldots, v_{ak}$. In the resulting election, $p$ ties $r$ in their head-to-head contest and $p$ beats every other candidate in their head-to-head contests. It follows that $p$ is a winner.

To prove the second statement, suppose that there is a subset $W$ of at most $k$ voters such that $r$ is not a unique winner of $E = (C, V - W)$. It is easy to see that $r$ beats every candidate in $\{\hat{r}, b_1, \ldots, b_{3k}\}$ in their head-to-head contests in $\hat{E}$. So, it certainly cannot be the case that $r$ beats $p$ in their head-to-head contest in $\hat{E}$. It follows that $\|W\| = k$ and that every voter in $W$ prefers $r$ to $p$. Note that both $r$ and $p$ beat $\hat{r}$ in their head-to-head contest in $\hat{E}$ and that both $r$ and $\hat{r}$ beat every $b_i \in B$ in their head-to-head contests in $\hat{E}$. It follows that the only possible winners in $\hat{E}$ are $r$ and $p$. (Note that without $\hat{r}$, it would be possible that after deleting $k$ voters, some $b_i$ beats all candidates other than $r$ in their head-to-head contests. If $\alpha = 0$, this could prevent $r$ from being the unique winner without necessarily making $p$ a winner.)

Let $b_i \in B$. Recall that $vs_E(b_i, p) = k - 4$ and that $p$ needs to beat $b_i$ in their head-to-head contest in $\hat{E}$. Since $\|W\| = k$, it follows that $W$ can contain at most one voter that prefers $p$ to $b_i$. Since $k > 2$ and every voter in $W$ prefers $r$ to $p$, it follows that $W$ contains only voters from the set $\{v_1, \ldots, v_n\}$ and that the voters in $W$ correspond to a $k$-element cover of $B$.

\[\square\]

**Theorem 4.27** Let $\alpha$ be a rational number such that $0 \leq \alpha \leq 1$. Copeland$^\alpha$ is resistant to both constructive and destructive control via partitioning voters in the TP model, in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.

**Proof.** Let $(B, S)$ be an instance of X3C, where $B = \{b_1, \ldots, b_{3k}\}$ and $S = \{S_1, \ldots, S_n\}$ is a finite family of three-element subsets of $B$. Without loss of generality, we assume that $n \geq k$ and that $k > 2$ (if $n < k$ then $S$ does not contain a cover of $B$, and if $k \leq 2$ then we can solve the problem by brute force). We build an election $E = (C, V)$ such that:

1. If $S$ contains a $k$-element cover of $B$, then the preferred candidate $p$ can become the unique Copeland$^\alpha$ winner of $E$ via partitioning voters in the TP model, and

2. if $r$ can be made to not uniquely win $E$ via partitioning voters in the TP model, then $S$ contains a $k$-element cover of $B$.
Note that this implies that Copeland$^\alpha$ is resistant to both constructive and destructive control via partitioning voters in the TP model, in both the nonunique-winner model and the unique-winner model.

Our construction is an extension of the construction from Theorem 4.25. We let the candidate set $C$ be $\{p, r, s, b_1, \ldots, b_{3k}\}$ and we let $V$ be the following set of voters:

1. We have $k + 1$ voters with preference $s > r > B > p$;
2. we have $n - 1$ voters with preference $B > p > r > s$;
3. we have $n - k + 2$ voters with preference $p > r > B > s$;
4. for each $S_i \in S$ we have two voters, $v_i$ and $v'_i$, such that
   (a) $v_i$ has preference $r > B - S_i > p > S_i > s$;
   (b) $v'_i$ has preference $r > S_i > p > B - S_i > s$.

Let $\hat{V} \subseteq V$ be the set of all the voters in $V$ except for the $k + 1$ voters with preference $s > r > B > p$. Note that $\hat{V}$ is exactly the set of voters used in the proof of Theorem 4.25 with candidate $s$ added as the least desirable candidate. Since $s$ does not influence the differences between the scores of the other candidates, the following claim follows immediately from the proof of Theorem 4.25.

**Claim 4.28** If $r$ can become a non-winner of $(C, \hat{V})$ by deleting at most $k$ voters, then $S$ contains a $k$-element cover of $B$.

Recall that we need to prove that if $S$ contains a $k$-element cover of $B$, then $p$ can be made the unique Copeland$^\alpha$ winner of $E$ via partitioning voters in the TP model, and that if $r$ can be made to not uniquely win $E$ via partitioning voters in the TP model, then $S$ contains a $k$-element cover of $B$.

If $S$ contains a $k$-element cover of $B$, say $\{S_{a_1}, \ldots, S_{a_k}\}$, then we let the second subelection consist of the $k + 1$ voters with preference $s > r > B > p$ and voters $v_{a_1}, \ldots, v_{a_k}$. Then $p$ is the unique winner of the first subelection, $s$ is the unique winner of the second subelection, and $p$ uniquely wins the final run-off between $p$ and $s$.

To prove the second statement, suppose there is a partition of voters such that $r$ is not a unique winner of the resulting election in model TP. Note that in at least one of the subelections, say the second subelection, a majority of the voters prefers $r$ to all candidates in $\{p, b_1, \ldots, b_{3k}\}$. Since $r$ is the unique winner of every run-off he or she participates in, $r$ cannot be a winner of either subelection. Since $r$ beats every candidate in $\{p, b_1, \ldots, b_{3k}\}$ in their head-to-head contests in the second subelection, in order for $r$ not to be a winner of the second subelection, it must certainly be the case that $s$ beats $r$ in their head-to-head contest in the second subelection. This implies that at most $k$ voters from $\hat{V}$ can be part of the second subelection.
Now consider the first subelection. Note that $r$ cannot be a winner of the first subelection. Then, clearly, $r$ cannot be a winner of the first subelection restricted to voters in $\tilde{V}$. By Claim 4.28 it follows that $S$ contains a $k$-element cover of $B$. 

**Theorem 4.29** Let $\alpha$ be a rational number such that $0 \leq \alpha < 1$. Copeland$^\alpha$ is resistant to constructive control via partitioning voters in the TE model, in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.

**Proof.** We use the exact same construction as in the proof of Theorem 4.27. We will show that if $S$ contains a $k$-element cover of $B$, then $p$ can be made the unique Copeland$^\alpha$ winner of $E$ via partitioning voters in the TE model and that if $p$ can be made a winner by partitioning voters in the TE model, then $S$ contains a $k$-element cover of $B$.

If $S$ contains a $k$-element cover of $B$, say $\{S_{a_1}, \ldots, S_{a_k}\}$, then we let the second subelection consist of the $k + 1$ voters with preference $s > r > B > p$ and voters $v_{a_1}, \ldots, v_{a_k}$. Then $p$ is the unique winner of the first subelection, $s$ is the unique winner of the second subelection, and $p$ uniquely wins the final run-off between $p$ and $s$.

To prove the second statement, suppose there is a partition of voters such that $p$ is a Copeland$^\alpha$ winner of the resulting election in model TE. Note that in at least one of the subelections, say the second subelection, a majority of the voters prefers $r$ to all candidates in $\{p, b_1, \ldots, b_{3k}\}$. Since $r$ is the unique winner of every run-off he or she participates in, $r$ can certainly not be the unique winner of the second subelection. Since $r$ beats every candidate in $\{p, b_1, \ldots, b_{3k}\}$ in their head-to-head contests in the second subelection, and since $s$ does not influence the relative scores of the candidates in $\{p, r, b_1, \ldots, b_{3k}\}$, no candidate in $\{p, b_1, \ldots, b_{3k}\}$ is a winner of the second subelection. It follows that $s$ is a winner of the second subelection. Since $\alpha < 1$, it follows that $s$ beats $r$ in their head-to-head contest in the second subelection. This implies that at most $k$ voters from $\tilde{V}$ can be part of the second subelection.

Now consider the first subelection. Note that $p$ must be the unique winner of the first subelection. So, certainly, $r$ cannot be a winner of the first subelection. Then, clearly, $r$ cannot be a winner of the first subelection restricted to voters in $\tilde{V}$. By Claim 4.28 it follows that $S$ contains a $k$-element cover of $B$. 

**Theorem 4.30** Copeland$^1$ is resistant to both constructive and destructive control via partitioning voters in the TE model, in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.

**Proof.** We use the same construction as in the proof of Theorem 4.27, except that we have one fewer voter with preference $s > r > B > p$. We will show that if $S$ contains a $k$-element cover of $B$, then $p$ can be made the unique Copeland$^1$ winner of $E$ via partitioning voters in the TE model and that if $r$ can be made to not uniquely win $E$ by partitioning voters in the TE model, then $S$ contains a $k$-element cover of $B$. 

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If \( S \) contains a \( k \)-element cover of \( B \), say \( \{S_{a_1}, \ldots, S_{a_k}\} \), then we let the second subelection consist of the \( k \) voters with preference \( s > r > B > p \) and voters \( v_{a_1}, \ldots, v_{a_k} \). Then \( p \) is the unique winner of the first subelection and proceeds to the run-off, and \( r \) and \( s \) are winners of the second subelection, and so no candidate from the second election proceeds to the run-off. It follows that \( p \) is the only candidate participating in the final run-off, and so \( p \) is the unique winner of the election.

To prove the second statement, suppose there is a partition of voters such that \( r \) is not a unique winner of the resulting election in model TE. Note that in at least one of the subelections, say the second subelection, a majority of the voters prefers \( r \) to all candidates in \( \{p, b_1, \ldots, b_{3k}\} \). Since \( r \) is the unique winner of every run-off he or she participates in, \( r \) should not participate in the final run-off. In particular, \( r \) cannot be the unique winner of the second subelection. The only way to avoid this is if \( r \) does not beat \( s \) in their head-to-head contest in the second subelection. This implies that at most \( k \) voters from \( \hat{V} \) can be part of the second subelection.

Now consider the first subelection. Note that \( r \) cannot be a winner of this subelection. Then, clearly, \( r \) cannot be a winner of this subelection restricted to voters in \( \hat{V} \). By Claim 4.28 it follows that \( S \) contains a \( k \)-element cover of \( B \). \( \square \)

**Theorem 4.31** Let \( \alpha \) be a rational number such that \( 0 \leq \alpha < 1 \). Copeland\(^\alpha \) is resistant to destructive control via partitioning voters in the TE model, in both the nonunique-winner model and the unique-winner model, for each in both the rational and irrational voter model.

**Proof.** Let \( (B, S) \) be an instance of X3C, where \( B = \{b_1, \ldots, b_{3k}\} \) and \( S = \{S_1, \ldots, S_n\} \) is a finite family of three-element subsets of \( B \). Without loss of generality, we assume that \( n \geq k \) and that \( k > 2 \) (if \( n < k \) then \( S \) does not contain a cover of \( B \), and if \( k \leq 2 \) then we can solve the problem by brute force). We build an election \( E = (C, V) \) such that:

1. If \( S \) contains a \( k \)-element cover of \( B \), then \( r \) can become a non-winner of \( E \) via partitioning voters in the TE model, and
2. if \( r \) can be made to not uniquely win \( E \) via partitioning voters in the TE model, then \( S \) contains a \( k \)-element cover of \( B \).

Note that this implies that Copeland\(^\alpha \) is resistant to destructive control via partitioning voters in the TE model, in both the nonunique-winner model and the unique-winner model.

In the proof of Theorem 4.27, we extended the construction from Theorem 4.25. In the proof of this theorem, we extend the construction from Theorem 4.26 in the same way.

We let the candidate set \( C \) be \( \{p, r, \hat{r}, s, b_1, \ldots, b_{3k}\} \) and we let \( V \) be the following set of voters:

1. We have \( k + 1 \) voters with preference \( s > r > \hat{r} > B > p \);
2. we have \( n - 2 \) voters with preference \( B > p > r > \hat{r} > s \);
3. we have \( n - k + 2 \) voters with preference \( p > r > \hat{r} > B > s \);

4. for each \( S_i \in S \) we have two voters, \( v_i \) and \( v'_i \), such that
   
   (a) \( v_i \) has preference \( r > \hat{r} > B - S_i > p > S_i > s \);
   
   (b) \( v'_i \) has preference \( r > \hat{r} > S_i > p > B - S_i > s \);

5. We have one voter with preference \( r > p > \hat{r} > B > s \);

6. We have one voter with preference \( B > p > r > \hat{r} > s \).

Let \( \hat{V} \subseteq V \) be the set of all the voters in \( V \) except for the \( k + 1 \) voters with preference \( s > r > \hat{r} > B > p \). Note that \( \hat{V} \) is exactly the set of voters used in the proof of Theorem 4.26 with candidate \( s \) added as the least desirable candidate. Since \( s \) does not influence the differences between the scores of the other candidates, the following claim follows immediately from the proof of Theorem 4.26.

Claim 4.32 If \( r \) can be made to not uniquely win \((C, \hat{V})\) by deleting at most \( k \) voters, then \( S \) contains a \( k \)-element cover of \( B \).

If \( S \) contains a \( k \)-element cover of \( B \), say \( \{S_{a_1}, \ldots, S_{a_k}\} \), then we let the second subelection consist of the \( k + 1 \) voters with preference \( s > r > \hat{r} > B > p \) and voters \( v_{a_1}, \ldots, v_{a_k} \). Then \( p \) is a winner of the first subelection, \( s \) is the unique winner of the second subelection, and it follows that \( r \) does not participate in the run-off.

For the second statement, suppose there is a partition of voters such that \( r \) is not a unique winner of the resulting election in model TE. Since \( r \) uniquely wins any run-off he or she participates in, it follows that \( r \) does not uniquely win either subelection. Note that in at least one of the subelections, say the second subelection, a majority of the voters prefers \( r \) to \( \hat{r} \) and \( r \) and \( \hat{r} \) to all candidates in \( \{p, b_1, \ldots, b_{3k}\} \). If \( s \) were to tie \( r \) in their head-to-head contest in the second subelection, then \( s \) would tie all candidates in the second subelection, and \( r \) would be the unique winner of the second subelection. It follows that \( s \) beats \( r \) in their head-to-head contest in the second subelection. This implies that at most \( k \) voters from \( \hat{V} \) can be part of the second subelection.

Now consider the first subelection. Note that \( r \) cannot be the unique winner of the first subelection. Then, clearly, \( r \) cannot be the unique winner of the first subelection restricted to voters in \( \hat{V} \). By Claim 4.32 it follows that \( S \) contains a \( k \)-element cover of \( B \). □

4.3 FPT Algorithm Schemes for Bounded-Case Control

Resistance to control is generally viewed as a desirable property in system design. However, suppose one is trying to solve resistant control problems. Is there any hope?

In their seminal paper on NP-hard winner-determination problems, Bartholdi, Tovey, and Trick [BTT89b] suggested considering hard election problems for the cases of a bounded number of candidates or a bounded number of voters, and they obtained efficient-algorithm
results for such cases. Within the study of elections, this same approach—seeking efficient fixed-parameter algorithms—has, for example, also been used (although somewhat tacitly—see the coming discussion in the second paragraph of Footnote 18) within the study of bribery [FHH06a,FHH06b]. To the best of our knowledge, this bounded-case approach to finding the limits of resistance results has not been previously used to study control problems. In this section we do precisely that.

In particular, we obtain for resistant-in-general control problems a broad range of efficient algorithms for the case when the number of candidates or voters is bounded. Our algorithms are not merely polynomial time. Rather, we give algorithms that prove membership in FPT (fixed-parameter tractability, i.e., the problem is not merely individually in P for each fixed value of the parameter of interest (voters or candidates), but indeed has a single P algorithm having degree that is bounded independently of the value of the fixed number of voters or candidates) when the number of candidates is bounded, and also when the number of voters is bounded. And we prove that our FPT claims hold even under the succinct input model—in which the voters are input via “(preference-list, binary-integer-giving-frequency-of-that-preference-list)” pairs—and even in the case of irrational voters.

We obtain such algorithms for all the voter-control cases, both for bounded candidates and for bounded voters, and for all the candidate-control cases with bounded candidates. On the other hand, we show that for the resistant-in-general irrational-voter, candidate-control cases, resistance still holds even if the number of voters is limited to being at most two.

We structure this section as follows. We first start by briefly stating our notions and notations. We next state, and then prove, our fixed-parameter tractability results. Regarding those, we first address FPT results for the (standard) constructive and destructive cases. We then show that in many cases we can assert FPT results that are more general still—in particular, we will look at “extended control”: completely pinpointing whether under a given type of control we can ensure that at least one of a specified collection of “Copeland Outcome Tables” (to be defined later) can be obtained. Finally, we give our resistance results.

Notions and Notations

The study of fixed-parameter complexity (see, e.g., [DF99,FG06]) has been expanding explosively since it was parented as a field by Downey, Fellows, and others in the late 1980s and the 1990s. Although the area has built a rich variety of complexity classes regarding parameterized problems, for the purpose of the current paper we need focus only on one very important class, namely, the class FPT. Briefly put, a problem parameterized by some value $j$ is said to be fixed-parameter tractable (equivalently, to belong to the class FPT) if there is an algorithm for the problem whose running time is $f(j)n^{O(1)}$. (Note in particular that there is some particular constant for the “big-oh” that holds for all inputs, regardless of what $k$ value the particular input has.)

In our context, we will consider two parameterizations: bounding the number of
candidates and bounding the number of voters. We will use the same notations used throughout this paper to describe problems, except we will postpend a “-BV\_j” to a problem name to state that the number of voters may be at most \( j \), and we will postpend a “-BC\_j” to a problem name to state that the number of candidates may be at most \( j \). In each case, the bound applies to the full number of such items involved in the problem. For example, in the case of control by adding voters, the \( j \) must bound the total of the number of voters in the election added together with the number of voters in the pool of voters available for adding.

Typically, we have been viewing input votes as coming in each on a ballot. However, one can also consider the case of succinct inputs, in which our algorithm is given the votes as “(preference-list, binary-integer-giving-frequency-of-that-preference-list)” pairs. (We mention in passing that for the “adding voter” cases, when we speak of succinctness we require that not just the always-voting voters be specified succinctly but also that the pool of voters-available-to-be-added be specified succinctly.) Succinct inputs have been studied extensively in the case of bribery \([\text{FHH06a}, \text{FHH06b}]\), and speaking more broadly, succinctness-of-input issues are often very germane to complexity classification (see, e.g., \([\text{Wag86}]\)). Note that proving an FPT result for the succinct case of a problem immediately implies an FPT result for the same problem (without the requirement of succinct inputs being in place), and indeed is a stronger result, since succinctness can potentially exponentially compress the input.

Finally, we would like to be able to concisely express many results in a single statement. To do so, we borrow a notational approach from transformational grammar, and use square brackets as an “independent choice” notation. So, for example, the claim \([\text{It, She, He, \[ runs, walks \]}\] is a shorthand for six assertions: It runs; She runs; He runs; It walks; She walks; and He walks. A special case is the symbol “\( \emptyset \)” which, when it appears in such a bracket, means that when unwound it should be viewed as no text at all. For example, “\([\text{Succinct, } \emptyset] \) Copeland is fun” asserts both “Succinct Copeland is fun” and “Copeland is fun.”

**Fixed-Parameter Tractability Results**

We immediately state our main results, which show that for all the voter-control cases FPT schemes hold for both the bounded-voter and bounded-candidate cases, and for all the candidate-control cases FPT schemes hold for the bounded-candidate cases.

**Theorem 4.33** For each rational \( \alpha \), \( 0 \leq \alpha \leq 1 \), and each choice from the independent choice brackets below, the specified parameterized (as \( j \) varies over \( \mathbb{N} \)) problem is in FPT:

\[
\begin{bmatrix}
\text{succinct} \\
\emptyset
\end{bmatrix}
\begin{bmatrix}
\text{Copeland}^\alpha \\
\text{Copeland}^\alpha_{\text{irrational}}
\end{bmatrix}
\begin{bmatrix}
\text{D} \\
\text{C}
\end{bmatrix}
\begin{bmatrix}
\text{AV} \\
\text{DV} \\
\text{PV-TE} \\
\text{PV-TP}
\end{bmatrix}
\begin{bmatrix}
\text{BV}_j \\
\text{BC}_j
\end{bmatrix}.
\]
Theorem 4.34  For each rational $\alpha$, $0 \leq \alpha \leq 1$, and each choice from the independent choice brackets below, the specified parameterized (as $j$ varies over $\mathbb{N}$) problem is in FPT:

\[
\begin{bmatrix}
\text{succinct} \\
\emptyset
\end{bmatrix} - \begin{bmatrix}
\text{Copeland}^\alpha \\
\text{Copeland}^\alpha_{\text{Irrational}}
\end{bmatrix} - \begin{bmatrix}
\text{AC}_u \\
\text{AC} \\
\text{DC} \\
\text{PC-TE} \\
\text{PC-TP} \\
\text{RPC-TE} \\
\text{RPC-TP}
\end{bmatrix} - \text{BC}_j.
\]

Readers not interested in a discussion of those results and their proofs can at this point safely skip to the next labeled section header.

Before proving the above theorems, let us first make a few observations about them. First, for cases where under a particular set of choices that same case is known (e.g., due to the results of Sections 4.1 and 4.2) to be in P even for the unbounded case, the above results are uninteresting as they follow from the earlier results (such cases do not include any of the “succinct” cases, since those were not treated earlier). However, that is a small minority of the cases. Also, for clarity as to what cases are covered, we have included some items that are not formally needed. For example, since FPT for the succinct case implies FPT for the no-succinctness-restriction case, and since FPT for the irrationality-allowed case implies FPT for the rational-only case, the first two choice brackets in each of the theorems could, without decreasing the results’ strength, be removed by eliminating their “$\emptyset$” and “Copeland$^\alpha$” choices.

We now turn to the proofs. Since proving every case would be uninterestingly repetitive, we will at times (after carefully warning the reader) prove the cases of one or two control types when that is enough to make clear how the omitted cases’ proofs go.

Let us start with those cases that can be done simply by appropriately applied brute force.

We first prove Theorem 4.34.

**Proof of Theorem 4.34.** If we are limited to having at most $j$ candidates, then for each of the cases mentioned, the total number of ways of adding/deleting/partitioning candidates is simply a (large) constant. For example, there will be at most ("at most" rather than "exactly" since $j$ is merely an upper bound on the number of candidates) $2^j$ possible run-off partitions and there will be at most $2^{j-1}$ relevant ways of deleting candidates (since we can’t (destructive case) or would never (constructive case) delete the distinguished candidate). So we can brute-force try all ways of adding/deleting/partitioning candidates, and for each such way can see whether we get the desired outcome. This works in polynomial time (with a fixed degree independent of $j$ and $\alpha$) even in the succinct case, and even with irrationality allowed.

A brute-force approach similarly works for the case of voter control when the number of voters is fixed. In particular, we prove the following subcase of Theorem 4.33.
Lemma 4.35 For each rational $\alpha$, $0 \leq \alpha \leq 1$, and each choice from the independent choice brackets below, the specified parameterized (as $j$ varies over $\mathbb{N}$) problem is in FPT:

$$\begin{bmatrix} \text{succinct} \\
\emptyset \end{bmatrix} - \begin{bmatrix} \text{Copeland}^\alpha \\
\text{Copeland}^\alpha_{\text{irrational}} \end{bmatrix} - \begin{bmatrix} C \\
D \\
PV-TE \\
PV-TP \end{bmatrix} - \text{BV}_j.$$

When considering “BV$_j$” cases—namely in this proof and in the resistance section starting on page 83—we will not even discuss succinctness. The reason is that if the number of voters is bounded, say by $j$, then succinctness doesn’t asymptotically change the input sizes interestingly, since succinctness at very best would compress the vote description by a factor of about $j$—which in this case is a fixed constant (relative to the value of the parameterization, which itself is $j$).

**Proof of Lemma 4.35.** If we are limited to having at most $j$ voters, note that we can, for each of these four types of control, brute-force check all possible approaches to that type of control. For example, for the case of control by deleting voters, we clearly have no more than $2^j$ possible vote deletion choices, and for the case of control by partitioning of voters, we again have at most $2^j$ partitions (into $V_1$ and $V - V_1$) to consider. And $2^j$ is just a (large) constant. So a direct brute-force check yields a polynomial-time algorithm, and by inspection one can see that its run-time’s degree is bounded above independently of $j$. $\square$

We now come to the interesting cluster of FPT cases: the voter-control cases when the number of candidates is bounded. Now, at first, one might think that we can handle this, just as the above cases, via a brute-force approach. And that is almost correct: One can get polynomial-time algorithms for these cases via a brute-force approach. However, for the succinct cases, the degrees of these algorithms will be huge, and will not be independent of the bound, $j$, on the number of candidates. For example, even in the rational case, one would from this approach obtain run-times with terms such as $n^{|C|^j}$. That is, one would obtain a family of P-time algorithms, but one would not have an FPT algorithm.

To overcome this obstacle, we will employ Lenstra’s [Len83] algorithm for bounded-variable-cardinality integer programming. Although Lenstra’s algorithm is truly amazing in its power, even it will not be enough to accomplish our goal. Rather, we will use a scheme that involves a fixed (though very large) number of Lenstra-type programs each being focused on a different resolution path regarding the given problem.

What we need to prove, to complete the proof of Theorem 4.33, is the following lemma.

Lemma 4.36 For each rational $\alpha$, $0 \leq \alpha \leq 1$, and each choice from the independent choice brackets below, the specified parameterized (as $j$ varies over $\mathbb{N}$) problem is in FPT:

$$\begin{bmatrix} \text{succinct} \\
\emptyset \end{bmatrix} - \begin{bmatrix} \text{Copeland}^\alpha \\
\text{Copeland}^\alpha_{\text{irrational}} \end{bmatrix} - \begin{bmatrix} C \\
D \\
PV-TE \\
PV-TP \end{bmatrix} - \text{BC}_j.$$
Let us start by recalling that, regarding the first choice bracket, the “succinct” case implies the “∅” case, so we need only address the succinct case. Recall also that, regarding the second choice bracket, for each rational \( \alpha \), \( 0 \leq \alpha \leq 1 \), the “\( \text{Copeland}^\alpha \) irrational” case implies the “\( \text{Copeland}^\alpha \)” case, so we need only address the \( \text{Copeland}^\alpha \) irrational case.

So all that remains is to handle each pair of choices from the third and forth choice brackets. To prove every case would be very repetitive. So we will simply prove in detail a difficult, relatively representative case, and then will for the other cases either mention the type of adjustment needed to obtain their proofs, or will simply leave it as a simple but tedious exercise that will be clear, as to how to do, to anyone who reads this section.

So, in particular, let us prove the following result.

**Lemma 4.37** For each rational \( \alpha \), \( 0 \leq \alpha \leq 1 \), the following parameterized (as \( j \) varies over \( \mathbb{N} \)) problem is in FPT: succinct-Copeland\(^\alpha\) irrational-CCPV-TP-BC\(_j\).

**Proof.** Let \( \alpha \), \( 0 \leq \alpha \leq 1 \), be some arbitrary, fixed rational number. In particular, suppose that \( \alpha \) can be expressed as \( b/d \), where \( b \in \mathbb{N} \), \( d \in \mathbb{N}^+ \), \( b \) and \( d \) share no common integer divisor greater than 1, and if \( b = 0 \) then \( d = 1 \). We won’t explicitly invoke \( b \) and \( d \) in our algorithm, but each time we speak of evaluating a certain set of pairwise outcomes “with respect to \( \alpha \),” one can think of it as evaluating that with respect to a strict pairwise win giving \( d \) points, a pairwise tie giving \( b \) points, and a strict pairwise loss giving 0 points.

We need a method of specifying the pairwise outcomes among a set of candidates. To do this, we will use the notion of a Copeland outcome table over a set of candidates. This will not actually be a table, but rather will be a function (a symmetric one—it will not be affected by the order of its two arguments) that, when given a pair of distinct candidates as inputs, will say which of the three possible outcomes allegedly happened: Either there is a tie, or one candidate won, or the other candidate won. So, in a \( j \)-candidate election, there are exactly \( 3^j \) such functions. (We will not care about the names of the candidates, and so will assume that the tables simply use the names 1 through \( j \), and that we match the names of the actual candidates with those integers by linking them lexicographically, i.e., the lexicographically first candidate will be associated with the integer 1 and so on.) Let us call a \( j \)-candidates Copeland outcome table a \( j \)-COT.

We need to build our algorithm that shows that the problem succinct-Copeland\(^\alpha\) irrational-CCPV-TP-BC\(_j\), \( j \in \mathbb{N} \), is in FPT. So, let \( j \) be some fixed integer bound on the number of candidates.\(^{18}\)

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\(^{18}\) We will now seem to specify the algorithm merely for this bound. However, it is important to note that we do enough to establish that there exists a single algorithm that fulfills the requirements of the definition of FPT. In particular, the specification we are about to give is sufficiently uniform that one can simply consider a single algorithm that, on a given input, notes the value of \( j \), the number of candidates, and then does what the “\( j \)’” algorithm we are about to specify does.

We take this moment to mention in passing that our earlier work, [FHH06a] and (this is an expanded, full version of that) [FHH06b], that gives P-time algorithms for the fixed parameter (fixed candidate and fixed voters) cases in fact, in all such claims we have in that work, implicitly is giving FPT algorithms, even although those papers don’t explicitly note that. The reason is generally the same as why that is true in this paper—namely, the Lenstra technique is not just powerful but is also ideally suited for FPT.
For each \( j'\)-COT, \( T_1 \),
For each \( j'\)-COT, \( T_2 \),
Do

If

when we have a Copeland\(^{\alpha}_{\text{Irrational}}\) election (involving all the input voters),
with respect to \( \alpha \), between all the candidates who win under \( T_1 \) with respect to \( \alpha \), and all the candidates who win under \( T_2 \) with respect to \( \alpha \), the preferred candidate of the input problem is a winner,

then

create and run the integer linear program constraint feasibility problem that checks whether there exists a partition of the voters such that the first subelection has \( j'\)-COT \( T_1 \) and the second subelection has \( j'\)-COT \( T_2 \), and if so, then accept.

Figure 7: The top-level code for the case succinct-Copeland\(^{\alpha}_{\text{Irrational}}\)-CCPV-TP-BC\(_j\)

Let us suppose we are given an input instance. Let \( j' \leq j \) be the number of candidates in this instance (recall that \( j \) is not the number of candidates, but rather is a bound on the number of candidates).

The top level of our algorithm is specified by the pseudocode in Figure 7. (Although this algorithm seemingly is just trying to tell whether the given control is possible for the given case, rather than telling how to partition to achieve that control, note that which iteration through the double loop accepts and the precise values of the variables inside the integer linear program constraint feasibility problem that made that iteration be satisfied will in fact tell us precisely what the partition is that makes the preferred candidate win.)

Now, note that the total number of \( j'\)-COTs that exist (we do not need to care whether all can be realized via actual votes) is \( 3^{\binom{j'}{2}} \). So the code inside the two loops executes at most \( 9^{\binom{j}{2}} \) times, which is constant-bounded since \( j' \leq j \), and we have fixed \( j \).

So all that remains is to give the integer linear program constraint feasibility problem mentioned inside the inner loop. The setting here can sometimes be confusing, e.g., when we speak of constants that can grow without limit. It is important to keep in mind that in this integer linear program constraint feasibility problem, the number of variables and algorithms and for being used inside algorithms that are FPT algorithms. Most interestingly, the Lenstra approach tends to work even on succinct inputs, and so the FPT comment we made applies even to those results in our abovementioned earlier papers that are about the succinct-inputs case of fixed-number-of-candidates and fixed-number-of-voters claims. (The fixed-number-of-candidates and fixed-number-of-voters Dodgson winner/score work of Bartholdi, Tovey, and Trick [BTT89b] is known to be about FPT algorithms (see [Bet07]). Although the paper of Bartholdi, Tovey, and Trick [BTT89b] doesn’t address the succinct input model, [FHH06a] notes that their approach works fine even in the succinct cases of the winner problem. That is true not just for the P-ness of their algorithms even in the succinct case, but also for the FPT-ness of their algorithms even in the succinct case.)
constraints is constant (over all inputs), and the integer linear program constraint feasibility problem’s “constants” are the only things that change with respect to the input. This is the framework that allows us to invoke Lenstra’s powerful algorithm.

We first specify the set of constants of the integer linear program constraint feasibility problem. In particular, for each \( i \), \( 1 \leq i \leq 2^{(j')} \), we will have a constant, \( n_i \), that is the number of input voters whose vote is of the \( i \)th type (among the \( 2^{(j')} \) possible vote possibilities; keep in mind that voters are allowed to be irrational, thus the value \( 2^{(j')} \) is correct). Note that the number of these constants that we have is itself constant-bounded (for fixed \( j \)), though of course the values that these constants (of the integer linear program constraint feasibility problem) take on can grow without limit.

In addition, let us define some constants that will not vary with the input but rather are simply a notational shorthand that we will use to describe how the integer linear program constraint feasibility problem is defined (what constraints occur in it). In particular, for each \( i \) and \( \ell \) such that \( 1 \leq i \leq j' \), \( 1 \leq \ell \leq j' \), and \( i \neq \ell \), let \( val1_{i,\ell} \) be 1 if \( T_1 \) asserts that (in their pairwise election) \( i \) ties or beats \( \ell \), and let it be 0 if \( T_1 \) asserts that (in their pairwise election) \( i \) loses to \( \ell \). Let \( val2_{i,\ell} \) be identically defined, except with respect to \( T_2 \). Informally put, these values will be used to let our integer linear program constraint feasibility problem seek to enforce such a win/loss/tie pattern with respect to the given input vote numbers and the given type of allowed control action.

The integer linear program constraint feasibility problem’s variables, which of course are all integer variables, are the following \( 2^{(j')} \) variables. For each \( i \), \( 1 \leq i \leq 2^{(j')} \), we will have a variable, \( m_i \), that represents how many of the \( n_i \) voters having the \( i \)th among the \( 2^{(j')} \) possible vote types go into the first subelection.

Finally, we must specify the constraints of our integer linear program constraint feasibility problem. We will have three groups of constraints.

The first constraint group is enforcing that plausible numbers are put in the first partition. In particular, for each \( i \), \( 1 \leq i \leq 2^{(j')} \), we have the constraints \( 0 \leq m_i \) and \( m_i \leq n_i \).

The second constraint group is enforcing that after the partitioning we really do have in the first subelection a situation in which all of the pairwise contests come out exactly as specified by \( T_1 \). In particular, for each \( i \) and \( \ell \) such that \( 1 \leq i \leq j' \), \( 1 \leq \ell \leq j' \), and \( i \neq \ell \), we do the following. Consider the equation

\[
( \sum_{\{a \mid 1 \leq a \leq 2^{(j')} \text{ and in votes of type } a \text{ it holds that } i \text{ is preferred to } \ell\}} m_a ) \text{ OP } ( \sum_{\{a \mid 1 \leq a \leq 2^{(j')} \text{ and in votes of type } a \text{ it holds that } \ell \text{ is preferred to } i\}} m_a ), \tag{4.a}
\]

where \( a \) in each sum varies over the \( 2^{(j')} \) possible preferences. If \( val1(i, \ell) = 1 \) we will have a constraint of the above form with OP set to “\( \geq \)”. If \( val1(\ell, i) = 1 \) we will have a constraint of the above form with OP set to “\( \leq \)”. Note that this means that if \( val1(i, \ell) = val1(\ell, i) = 1 \),
i.e., those two voters are purported to tie, we will add two constraints.

The third constraint group has the same function as the second constraint group, except it regards the second subelection rather than the first subelection. In particular, for each $i$ and $\ell$ such that $1 \leq i \leq j'$, $1 \leq \ell \leq j'$, and $i \neq \ell$, we do the following. Consider again equation 4.a from above, except with each of the two occurrences of $m_a$ replaced by $n_a - m_a$. If \( \text{val}_2(i, \ell) = 1 \) we will have a constraint of that form with \( \text{OP} \) set to \( \geq \). If \( \text{val}_2(\ell, i) = 1 \) we will have a constraint of that form with \( \text{OP} \) set to \( \leq \). As above, this means that if \( \text{val}_2(i, \ell) = \text{val}_2(\ell, i) = 1 \), we will add two constraints.

This completes the specification of the integer linear programming constraint feasibility problem.

Note that our top-level code, from Figure 7, clearly runs within polynomial time relative to even the succinct-case input to the original CCP V-TP problem, and that polynomial’s degree is bounded above independently of $j$. Note in particular that our algorithm constructs at most a large constant (for $j$ fixed) number of integer linear programming constraint feasibility problems, and each of those is itself polynomial-sized relative to even the succinct-case input to the original CCP V-TP problem, and that polynomial size’s degree is bounded above independently of $j$. Further, note that the integer linear programming constraint feasibility problems clearly do test what they are supposed to test—most importantly, they test that the subelections match the pairwise outcomes specified by $j'$-COTs $T_1$ and $T_2$. Finally and crucially, by Lenstra’s algorithm ([Len83], see also [Dow03,Nie02] which are very clear regarding the “linear’s” later in this sentence), since this integer linear programming constraint feasibility problem has a fixed number of constraints (and in our case in fact also has a fixed number of variables), it can be solved—relative to its size (which includes the filled-in constants, such as our $n_i$, for example, which are in effect inputs to the integer program’s specification)—via a linear number of arithmetic operations on linear-sized integers. So, overall, we are in polynomial time even relative to succinctly specified input, and the polynomial’s degree is bounded above independently of $j$. Thus we have established membership in the class FPT.

We now describe very briefly how the above proof of Lemma 4.37 can be adjusted to handle all the partition cases from Lemma 4.36, namely, the cases $[\text{succinct }] = \left[ \begin{array}{c} \text{Copeland}^{\alpha} \\ \text{Copeland}_{\text{Irrational}}^{\alpha} \end{array} \right]$, $[C]$, $[D]$, $[\text{PV-TE}]$, $[\text{PV-TP}]$, $\text{-BC}_j$. As noted before, the first two brackets can be ignored, as we have chosen the more demanding choice for each. Let us discuss the other variations. Regarding changing from constructive to destructive, in Figure 7 change “is a winner” to “is not a winner.” Regarding changing from PV-TP to PV-TE, in the “if” block in Figure 7 change each “all the candidates who win” to “the candidate who wins (if there is a unique candidate who wins).”

The only remaining cases are the cases $[\text{succinct }] = \left[ \begin{array}{c} \text{Copeland}^{\alpha} \\ \text{Copeland}_{\text{Irrational}}^{\alpha} \end{array} \right]$, $[C]$, $[D]$, $[\text{AV}]$, $[\text{DV}]$, $\text{-BC}_j$. However, these cases are even more straightforward than the partition cases we just covered, so for space reasons we will not write them out, but rather will briefly comment on these cases. Basically, one’s top-level code for these cases loops over all $j'$-COTs, and for each
(there are \(3^{\binom{j}{2}}\)) checks whether the right outcome happens under that \(j'\)-COT (i.e., the distinguished candidate either is (constructive case) or is not (destructive case) a winner), and if so, it runs Lenstra’s algorithm on an integer linear programming constraint feasibility problem to see whether we can by the allowed action (adding/deleting) get to a state where that particular \(j'\)-COT matches our (after addition or deletion of voters) election. In the integer program, the variables will be the obvious ones, namely, for each \(i\), \(1 \leq i \leq 2^{\binom{j}{2}}\), we will have a variable, \(m_i\), that describes how many voters of type \(i\) to add/delete. As our key constants (of the integer linear program constraint feasibility problem), we will have, for each \(i\), \(1 \leq i \leq 2^{\binom{j}{2}}\), a value, \(n_i\), for the number of type \(i\) voters in the input. Also, if this is a problem about addition of voters, we will have additional constants, \(\hat{n}_i\), \(1 \leq i \leq 2^{\binom{j}{2}}\), representing the number of type \(i\) voters among the pool, \(W\), of voters available for addition. And if our problem has an internal “\(k\)” (a limit on the number of additions or deletions), we enforce that with the natural constraints, as do we also with the natural constraints enforce the obvious relationships between the \(m_i\), \(n_i\), \(\hat{n}_i\), and so on. Most critically, we have constraints ensuring that after the additions/deletions specified by the \(m_i\) each pairwise outcome specified by the \(j'\)-COT is realized.

Finally, although everything in Section 4.3 (both the part so far and the part to come) is written for the case of the nonunique-winner model, all the results hold analogously in the unique-winner model, with the natural, minor proof modifications. (Also, we mention in passing that due to the connection, found in Footnote 5 of [HHR07a], between unique-winner destructive control and nonunique-winner constructive control, one could use some of our nonunique-winner constructive-case results to indirectly prove some of the unique-winner destructive-case results.)

**FPT and Extended Control**

In this section, we look at extended control. By that we do not mean changing the ten standard control notions of adding/deleting/partitioning candidates/voters. Rather, we mean generalizing past merely looking at the constructive (make a distinguished candidate a winner) and the destructive (prevent a distinguished candidate from being a winner) cases. In particular, we are interested in control where the goal can be far more flexibly specified, for example (though in the partition cases we will be even more flexible than this), we will allow as our goal region any (reasonable—there are some time-related conditions) subcollection of “Copeland outcome tables” (specifications of who won/lost/tied each head-to-head contest). Since from a Copeland outcome table, in concert with the current \(\alpha\), one can read off the Copeland\(^{\alpha}\)\(_{\text{Irrational}}\) scores of the candidates, this allows us a tremendous range of descriptive flexibility in specifying our control goals, e.g., we can specify a linear order desired for the candidates with respect to their Copeland\(^{\alpha}\)\(_{\text{Irrational}}\) scores, we can specify a linear-order-with-ties desired for the candidates with respect to their Copeland\(^{\alpha}\)\(_{\text{Irrational}}\) scores, we can specify the exact desired Copeland\(^{\alpha}\)\(_{\text{Irrational}}\) scores for one or more candidates, we can specify that we want to ensure that no candidate from a certain subgroup has a...
Copeland\textsuperscript{a} \textsubscript{Irrational} score that ties or beats the Copeland\textsuperscript{a} \textsubscript{Irrational} score of any candidate from a certain other subgroup, etc.\textsuperscript{19} Later in this section we will give a list repeating some of these examples and adding some new examples.

All the FPT algorithms given in the previous section regard, on their surface, the standard control problem, which tests whether a given candidate can be made a winner (constructive case) or can be precluded from being a winner (destructive case). We now note that the general approaches used in that section in fact yield FPT schemes even for the far more flexible notions of control mentioned above. In fact, one gets, for all the FPT cases covered in the previous section, FPT algorithms for the extended-control problem for those cases—very loosely put, FPT algorithms that test, for virtually any natural collection of outcome tables (as long as that collection itself can be recognized in a way that doesn’t take too much running time, i.e., the checking time is polynomial and of a degree that is bounded independently of \( j \)), whether by the given type of control one can reach one of those outcome tables.

Let us discuss this in a bit more detail. A key concept used inside the proof of Lemma 4.37 was that of a Copeland outcome table—a function that for each distinct pair of candidates specifies either a tie or specifies who is the (not tied) winner in their pairwise contest. Let us consider the control algorithm given in the proof of that lemma, and in particular let us consider the top-level code specified in Figure 7. That code double-loops over size \( j' \) Copeland outcome tables (a.k.a. \( j' \)-COTs), regarding the subpartitions, and for each case when the outcome tables’ subelection cases, followed by the final election that they imply, correspond to the desired type of constructive (the distinguished person wins) or destructive (the distinguished person does not win) outcome, we check whether those two \( j' \)-COTs can be made to hold via the current type of control (for the case being discussed, PV-TP).

However, note that simply by easily varying that top-level code we can obtain a natural FPT algorithm (a single algorithm, see Footnote 18 the analogue of which applies here) for any question of whether via the allowed type of control one can reach any run-time-quick-to-recognize collection of pairs of \( j' \)-COTs (in the subelection), or even whether a given candidate collection and one of a given (run-time-quick-to-recognize) \( j'' \)-COT collection over that candidate collection (\( j'' \) being the size of that final-round candidate collection) can be reached in the final election. This is true not just for the partition cases (where, informally put, we would do this by, in Figure 7, changing the condition inside the “if” to

\textsuperscript{19}We mention up front that that initial example list applies with some additional minor technical caveats. Those examples were speaking as if in the final election we have all the candidates receiving Copeland\textsuperscript{a} \textsubscript{Irrational} scores in the final election. But in fact in the partition cases this is not (necessarily) so, and so in those cases we will focus on the Copeland outcome tables most natural to the given case. For example, in control by partition of voters, we will focus on subcollections of pairs of Copeland outcome tables for the two subelections. Also, though our Copeland outcome tables as defined below are not explicitly labeled with candidate names, but rather use a lexicographical correspondence with the involved candidates, in some cases we would—though we don’t repeat this in the discussion below—need to allow the inclusion in the goal specification of the names of the candidates who are in play in a given table or tables, most particularly, in the cases of addition and deletion of candidates, and in some partition cases.
instead look for membership in that collection of \( j'-\text{COTs} \) but also for all the cases we attacked via Lenstra’s method (though for the nonpartition cases we will typically single-loop over Copeland outcome tables that may represent the outcome after control is exerted; also, for some of these cases, the caveat at the end of Footnote 19 will apply). And it is even easier to notice that for those cases we attacked by direct brute force this also holds.

So, as just a few examples (some echoing the start of this section, and some new), all the following have (with the caveats mentioned above about needed names attached, e.g., in cases of candidate addition/deletion/partition, and regarding the partition cases focusing not necessarily directly on the final table) FPT extended control algorithms for all the types of control and boundedness cases for which the FPT results of the previous section are stated.

1. Asking whether under the stated action one can obtain in the final election (simply in the election in the case when there is not partitioning) the outcome that all the Copeland\(^{a}\)\(^{\text{Irrational}}\)-system scores of the candidates precisely match the relations of the lexicographic names of the candidates.

2. More generally than that, asking whether under the stated action one can obtain in the final election (simply in the election in the case when there is not partitioning) a certain linear-order-without-ties regarding the Copeland\(^{a}\)\(^{\text{Irrational}}\)-system scores of the candidates.

3. More generally still, asking whether under the stated action one can obtain in the final election (simply in the election in the case when there is not partitioning) a certain linear-order-with-ties regarding the Copeland\(^{a}\)\(^{\text{Irrational}}\)-system scores of the candidates.

4. Asking whether under the stated action one can obtain in the final election (simply in the election in the case when there is not partitioning) the situation that exactly 1492 candidates tie as winner regarding their Copeland\(^{a}\)\(^{\text{Irrational}}\)-system scores.

Let us discuss this a bit more formally, again using PV-TP as an example. Consider any family of boolean functions \( F_j, j \in \mathbb{N} \), such that each \( F_j \) is computable, even when its first argument is succinctly specified, in polynomial time with the polynomial degree bounded independently of \( j \). Now, consider changing Figure 7’s code to:

For each \( j'-\text{COT}, T_1 \),
For each \( j'-\text{COT}, T_2 \),
If \( (F_j,(\text{input},T_1,T_2)) \) then · · · .

Note that this change gives an FPT control scheme for a certain extended control problem. In particular, it does so for the extended control problem whose goal is to ensure that we can realize at least one of the set of \( (T_1,T_2) \) such that \( F_j \) (\( j' \) being the number of candidates in the particular input), given as its inputs the problem’s input, \( T_1 \), and \( T_2 \) evaluates to true. That is, the \( F_j \) functions are recognizing (viewed a bit differently, are defining) the goal set of the extended control problem.

From the input, \( T_1 \), and \( T_2 \) we can easily tell the scores in the final election. So this approach can be used to choose as our extended-control goals natural features of the final election.
5. Asking whether under the stated action one can obtain in the final election (simply in the election in the case when there is not partitioning) the situation that no two candidates have the same Copeland\(^a_{\text{Irrational}}\)-system scores as each other.

Again, these are just a very few examples. Our point is that the previous section is flexible enough to address not just constructive/destructive control, but also to address far more general control issues.

### Resistance Results

Theorems 4.33 and 4.34 give FPT schemes for all voter-control cases with bounded voters, for all voter-control cases with bounded candidates, and for all candidate-control cases with bounded candidates. This might lead one to hope that all the cases admit FPT schemes. However, the remaining type of case, the candidate-control cases with bounded voters, does not follow this pattern. In fact, we note that for Copeland\(^a_{\text{Irrational}}\) all the candidate-control cases that we showed earlier in this paper (i.e., without bounds on the number of voters) to be resistant remain resistant even for the case of bounded voters. This resistance holds even when the input is not in succinct format, and so it certainly also holds when the input is in succinct format.

The reason for this is that, for the case of irrational voters, with just two voters (with preferences over \(j\) candidates) any given \(j\)-COT can be achieved. To do this, for each distinct pair of candidates \(i\) and \(\ell\), to have \(i\) preferred in their pairwise contest have both voters prefer \(i\) to \(\ell\), to have \(\ell\) preferred in their pairwise contest have both voters prefer \(\ell\) to \(i\), and to have a tie in the pairwise contest have one voter prefer \(\ell\) to \(i\) and one voter prefer \(i\) to \(\ell\). Then, since our earlier resistance results on these issues are proven by simple reductions (from a vertex cover problem) in which we carefully set the \(j\)-COT (or the linear order, but note that \(j\)-COTs can capture any linear order), it is not hard to see that all these resistance proofs carry over even to the case of two voters.

The only open cases remaining regard the rational-voter, candidate-control, bounded-voter cases.

### 5 Control in Condorcet Elections

In this section we show that Condorcet elections are resistant to constructive control via both deleting voters (CCDV) and partition of voters (CCPV). These results were originally claimed in the seminal paper of Bartholdi, Tovey, and Trick [BTT92], but the proofs therein were based on the assumption that a voter can be indifferent between several candidates. Strictly speaking, their model of election did not allow that (and neither does ours). Here we show how one can obtain these results in the case when the voters’ preference lists are linear orders.

A candidate \(c\) of election \(E = (C, V)\) is a Condorcet winner of \(E\) if he or she defeats all other candidates in their head-to-head contests. Alternatively, one could say that a candidate \(c\) is a Condorcet winner of election \(E\) if and only if he or she has Copeland\(^0\)
score of \( \|C\| - 1 \). Naturally, each election can have at most one Condorcet winner and thus it doesn’t make sense here to differentiate between the unique-winner and the nonunique-winner models.

**Theorem 5.1** Condorcet elections are resistant to constructive control via deleting voters (CCDV).

**Proof.** The result follows via a reduction from the X3C problem. Let \((B, S)\) be the input X3C instance, where \(B = \{b_1, \ldots, b_{3k}\}\) is some finite set and \(S = \{S_1, \ldots, S_n\}\) is some finite family of size-three subsets of \(B\). We assume that \(\bigcup_{i=1}^{n} S_i = B\), as otherwise the instance definitely is not a member of X3C and we can output a fixed negative instance \(S\) corresponding to those sets \(b\) exactly one vote. Let us fix an arbitrary candidate \(k\) —

Given \((B, S)\), we form Condorcet election \(E = (C, V)\) with \(C = \{p, r, b_1, \ldots, b_{3k}\}\) and \(4n - k + 1\) voters set in the following way. For every \(S_i \in S\), we have two voters, \(v_i\) and \(v_i'\), where \(v_i\)'s preference list is

\[ r > B - S_i > p > S_i, \]

and \(v_i'\)'s preference list is

\[ r > S_i > p > B - S_i. \]

We also have \(n - k + 2\) voters, \(u_1, \ldots, u_{n-k+2}\), who report the preference list \(p > r > B\) and \(n - 1\) voters, \(w_1, \ldots, w_{n-1}\), who report the preference list \(B > p > r\).

Via simple counting of votes it is easy to see that before deleting any voters \(r\) is the Condorcet winner and that \(r\) defeats \(p\) by exactly \(k - 1\) votes. Also, each of the \(b_i\)'s defeats \(p\) by exactly \(k - 3\) votes. We claim that \(p\) can become a Condorcet winner of this election if and only if \(S\) contains an exact cover of \(B\).

If there are \(k\) sets \(S_{a_1}, \ldots, S_{a_k}\) whose union is \(B\) (i.e., if \(S\) contains an exact cover of \(B\)) then we can make \(p\) a Condorcet winner of \(E\) via deleting exactly the voters \(v_{a_1}, \ldots, v_{a_k}\). Deleting these \(k\) voters guarantees that \(p\) wins his or her head-to-head contest with \(r\) by exactly one vote. Let us fix an arbitrary candidate \(b_i\) and see how deleting voters \(v_{a_1}, \ldots, v_{a_k}\) affects the result of \(b_i\)'s head-to-head contest against \(p\). Before deleting any voters, we have \(\text{vs}_E(b_i, p) = k - 3\), i.e., \(b_i\) has \(k - 3\) votes of advantage over \(p\). Deleting \(k - 1\) voters that correspond to those sets \(S_i\) from the cover that do not contain \(b_i\) decreases this value by \(k - 1\) and deleting the voter that corresponds to a set containing \(b_i\) increases it by 1. In the end, if \(E'\) is the election with the voters corresponding to the cover deleted, we have \(\text{vs}_{E'}(p, b_i) = 1\). That is, \(p\) wins by one vote. Thus, since \(b_i\) was chosen arbitrarily, after deleting voters \(v_{a_1}, \ldots, v_{a_k}\), \(p\) wins the head-to-head contests with all candidates \(b_1, \ldots, b_{3k}\), and since we have seen that then \(p\) also wins against \(r\), \(p\) is the Condorcet winner.

For the converse assume that \(p\) can become a Condorcet winner after deleting at most \(k\) voters, and let \(W\) be a subset of \(V\) such that (a) \(\|W\| \leq k\), and (b) deleting the voters in \(W\) from the election guarantees that \(p\) is a Condorcet winner. It is easy to see that \(W\) contains exactly \(k\) voters chosen from the set \(\{v_1, v_1', \ldots, v_n, v_n'\}\); these are the only voters
who prefer $r$ to $p$ and before any deletions $p$ loses his or her head-to-head contest with $r$ by $k - 1$ votes.

We also note that for each candidate $b_i$, $W$ may contain at most one voter that prefers $p$ to $b_i$. For the sake of contradiction let us fix some candidate $b_i$ and assume that more than one voter in $W$ prefers $p$ to $b_i$. However, recall that before any deletions, $p$ loses his or her head-to-head contest with $b_i$ by exactly $k - 3$ votes. If we delete at least two voters that prefer $p$ to $b_i$ then this number increases to at least $k - 1$ and it is impossible to bring it below zero via deleting at most $k - 2$ voters, even if they all prefer $b_i$ to $p$. Thus, $W$ contains exactly $k$ voters and for each $b_i$ only one of those $k$ voters prefers $p$ to $b_i$. This implies that $W$ corresponds to an exact cover of $B$ (recall that $k \geq 3$). This completes the proof.

Before we proceed with our proof of resistance for the case of constructive control via partition of voters (CCPV), we have to mention a slight quirk of Bartholdi, Tovey, and Trick’s model of voter partition. If one reads their paper carefully, it becomes apparent that they have a quiet assumption that each given set of voters can only be partitioned into subelections that each elect exactly one winner, thus severely restricting the chair’s partitioning possibilities. That was why Hemaspaandra, Hemaspaandra, and Rothe [HHR07a] replaced Bartholdi, Tovey, and Trick’s convention with the more natural ties-promote and ties-eliminate rules (see the discussion in [HHR07a], but within this section we go back to Bartholdi, Tovey, and Trick’s model, since our goal here is to reprove their results without breaking their model.

**Theorem 5.2** Condorcet elections are resistant to constructive control via partitioning voters (CCPV) in Bartholdi, Tovey, and Trick’s model (see the paragraph above).

**Proof.** The proof follows via a reduction from the $X3C$ problem. In fact, we use exactly the construction from Theorem 5.1 with the following two modifications:

1. We add a new candidate $w$ that all the voters mentioned in the proof of Theorem 5.1 rank last.

2. We add $k + 1$ voters, $q_1, \ldots, q_{k+1}$, each with preference list $w > B > r > p$.

   Since $w$ is the only candidate that $p$ defeats in a head-to-head contest, the only way for $p$ to become a winner via partitioning voters is to guarantee that $p$ wins within his or her subelection and that $w$ wins within the other one. (Note that since $p$ is not a Condorcet winner, he/she cannot win in both subelections.)

   Let $(V_p, V_w)$ be a partition of the collection of voters such that $p$ is the global winner if we use two subelections, one with voters $V_p$ and one with voters $V_w$. Via the above paragraph we can assume, without loss of generality, that $p$ is a Condorcet winner in $V_p$ and that $w$ is a Condorcet winner in $V_w$. Since each voter $q_i$, $1 \leq i \leq k + 1$, prefers $w$ to everyone else and prefers anyone else to $p$, we can assume that $V_w$ contains all of the voters $q_1, \ldots, q_{k+1}$. Also, $V_w$ contains at most $k$ voters other than $q_1, \ldots, q_{k+1}$, as otherwise $w$ would certainly not be a Condorcet winner in $V_w$.

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As a result, $V_p$ contains all voters except for $q_1, \ldots, q_{k+1}$ and at most $k$ others. However, as shown in the proof of Theorem 5.1, $p$ can be a Condorcet winner of such an election if and only if there is an exact cover of $B$ in $S$ (omitting of to $k$ voters from $V_p$ here corresponds exactly to deleting the voters in Theorem 5.1; also, note that aside from $q_1, \ldots, q_{k+1}$ all voters rank $w$ last, so his or her presence does not affect the logic of the proof of Theorem 5.1). This completes the proof.

6 Conclusions

We have shown that from the computational point of view the election systems of Llull and Copeland (i.e., Copeland$^{0.5}$) are broadly resistant to bribery and procedural control, regardless of whether the voters are required to have rational preferences. It is rather charming that Llull’s 700-year-old system shows perfect resistance to bribery and more resistances to (constructive) control than any other natural system (even far more modern ones) with an easy winner-determination procedure—other than the Copeland$^\alpha$, $0 < \alpha < 1$—is known to possess, and this is even more remarkable when one considers that Llull’s system was defined long before control of elections was even explicitly studied. Copeland voting matches Llull’s perfect resistance to bribery and in addition has perfect resistance to (constructive) control.

A natural open direction would be to study the complexity of control for additional election systems. Particularly interesting would be to seek existing, natural voting systems that have polynomial-time winner determination procedures but that are resistant to all standard types of both constructive and destructive control. Also extremely interesting would be to find single results that classify, for broad families of election systems, precisely what it is that makes control easy or hard, i.e., to obtain dichotomy meta-results for control (see Hemaspaanda and Hemaspaanda [HH07] for some discussion regarding work of that flavor for manipulation).

References


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