Essays on Stochastic Games and on Strategic Equivalence between Normal Form Games

by

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To Marcelle.
Curriculum Vitae

The author was born in Rio de Janeiro, Brazil, on February 13, 1984. He attended the Getulio Vargas Foundation’s School of Economics from 2001 to 2006, and graduated with a Bachelor of Arts in Economics in 2006. Before coming to the University of Rochester, he was an exchange student at the California Institute of Technology, where he studied with the Control and Dynamical Systems and the Economics Departments. He came to the University of Rochester in the Fall of 2006 and began his graduate studies in Economics. He received a University of Rochester fellowship from 2006 to 2010, an IBRE-FGV fellowship from 2006-2010, and a Wallis Fellowship both from 2006 to 2008 as well as from 2009 to 2010. He received his Master of Arts in Economics from the University of Rochester in 2009. Since then, he has been researching Game Theory under the supervision of Professor Paulo Barelli.
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Abstract

This thesis is a collection of essays on stochastic games with incomplete information and one essay on strategic equivalence between normal form games.

In Chapter 1, we propose a new class of stochastic games suitable to the study of situations where there is incomplete information about the payoff relevant state of the world. We show the existence of equilibrium in pure Markov strategies by applying a few restrictions on the framework. We also investigate when the Markov assumption on strategies is reasonable. We show that if the Markov law of motion on the state of the world has full support, players’ beliefs also have full support, players do not learn, and the Markov assumption is reasonable. Another type of situation in which we can guarantee that the Markov assumption is reasonable, is when the game becomes large, if players are interconnected and the dynamics on the state space implied by the Markov law of motion are rich. As an application, we study a dynamic arms race model under incomplete information, an imperfect market competition model where the privately observed variable may be serially correlated over time, and a dynamic search model with hidden search productivity.

In Chapter 2, we provide an application of the framework developed in Chapter 1 to multistage R&D races. The framework developed in Chapter 1 is suitable to this type of problem, because R&D races typically involve the development of several intermediate steps and in some industries information about the competitor’s development stage is strategic. In our R&D model, firms need to complete a finite number of intermediate steps before the product or project is finished. In our model, there is incomplete information because firms may not know the competitor’s development stage during the race. By using numerical methods, we study the
equilibrium of this race in pure Markov strategies. We show how the race duration, consumer surplus, firm value and total welfare vary with investment costs, market size and intensity of competition in the products market. We are able to assess the impact of a strong patent regime in terms of consumer surplus, firm value and total welfare.

In Chapter 3, we provide sufficient conditions for uniqueness of Bayesian Nash equilibrium on supermodular incomplete information games. This theorem is used both in Chapter 2 and Chapter 3, and given its more general interest, we include it separately in this chapter.

In Chapter 4, we study the problem of equivalence between finite normal form games. We introduce a procedure to associate an equivalence relation on the space of normal form games for a solution concept. This equivalence relation is able to compare games with different number of pure strategies. We show that this relation is indeed an equivalence relation, i.e. symmetric, reflexive and transitive. In addition, we show that the number of equivalence classes of an equivalence relation is countable in the space of finite normal form games. If there exists an upper bound on the number of pure strategies, the number of equivalence classes is finite. For the case where there exists a maximum number of pure strategies, we are able to find an upper bound on the number of equivalence classes. An application to the Nash equilibrium equivalence relation is provided.
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Chapter 1

On Stochastic Incomplete Information Games with Persistent Private Information

1.1 Introduction

The objective of this chapter is to define a class of dynamic games suitable to the study of strategic situations where players may not observe a payoff relevant state variable and its transition may be persistent over time.

In this class of games, there exists a state space that summarizes all payoff relevant information. Furthermore, at each period, the finite set of players may face uncertainty about the state of the world, because their only source of information is an event in $S$. That is, they observe a collection of states that they are sure the true state of the world is in. Also, the set of events each player observes may not be the same for all players. They do not have to be symmetrically informed during the course of the game. For example, we could have the private information case where each player only observes his own state variable. We could also have a complete information setup, where each player observes the state of the world and other types of information structures, such as the case where only one player knows exactly the state of the world while the others players do not.
The state of the world’s transition depend on players’ actions. At each period, players take an action and after they receive payoff, the next state of the world is drawn from a stochastic law of motion, which is Markov. We call the law of motion Markov because the probability of the next period’s state depends only on the actual period’s state and on players’ actions, not on a larger history of events. The state draws do not have to be iid across periods. Hence, we can study in our framework strategic situations where the privately informed state transition is serially correlated.

This type of framework is useful, for example, to study dynamic imperfect competition models. We can extend Pakes and Ericson’s model and allow scrap values and entry values to be serially correlated. With this possibility, their model becomes more realistic, because it is more intuitive that scrap values and entry costs do not change from one period to another in some unrelated way. The same argument can be made for imperfect competition models where the state variable are production costs. One example of such type of models is Athey and Bagwell. Other types of useful strategic situations include arms race, where each country may not know the other country’s arms level; R&D races, where firm’s are not perfectly informed about the other firms development stage, to cite a few.

As players may not be symmetrically informed about the state of the world, not only are their beliefs about the state of the world relevant in deciding the best action, but also their beliefs about the other players’ beliefs about the state of the world and so on. To model this interactive uncertainty we take the Bayesian approach and endow each player with a type from a dynamic type space (Battigalli and Siniscalchi). A dynamic type extends the usual notion of type proposed by Harsanyi and later formalized by Mertens and Zamir (among others)
to dynamic incomplete information games. In dynamic type spaces each player is characterized by a type and a family of observable events. The player’s beliefs are represented by a Conditional Probability System (Myerson [38]), which is a list of probability distributions, one for each event the player may observe, over the other players’ types and the state space. A Conditional Probability System is useful to deal with conditional probabilities, especially when an event has zero probability a priori and Bayesian updating is not well defined. In our framework, player’s beliefs about the current state of the world depends on their type and the event that they observe and it is always well defined.

The framework we described is too general to get an equilibrium existence result. Hence, to establish the existence of equilibrium we make some assumptions on the player’s payoff function, beliefs, and Markov law of motion. Apart from the assumption on beliefs, these assumptions are found elsewhere in the literature. They allow us to adapt the existing equilibrium existence proofs for stochastic complete information games to our incomplete information framework.

We require the player’s payoff function to be twice continuously differentiable, concave in the player’s own actions, have increasing differences and to satisfy strict diagonal dominance (Gabay and Moulin [18], Milgrom and Roberts [37]). In addition, the type space should be such that there is no irrelevant type. That is, there must always be some player’s type that puts positive probability on every type profile of other players’ types. We require the Markov law of motion to be a convex combination between two Markov chains over the state space.

In our main theorem we show the existence of equilibrium in pure Markov strategies. A pure Markov strategy is defined as a mapping from types and observable events into actions. We call the strategy Markov because it does not depend on
the history of observable events, only on the event that each player observes in that period. This type of strategy has a long tradition in dynamic games (Maskin and Tirole [27], [28] and [29]). According to Maskin and Tirole [30] it captures the idea of 'bygones are bygones' or that minor facts should have minor consequences.

The method of proof for the equilibrium existence theorem consists of two main steps. First, we show that the static incomplete information game induced at each state of the world given a value function which summarizes the expected discounted future payoff has a unique Bayesian Nash equilibrium. The result is an application of Polydoro [44] for this setup. In the second step, we show that there also exists a value function consistent with the Bayesian Nash equilibrium in each induced static incomplete information game. That is, given the equilibrium value function, the expected payoff at the best action for each player’s type at each observable event is equal to the value function for that type at that observable event. This step is based on the Nowak’s [39] existence proof for Markov perfect equilibrium in stochastic complete information games, which is in turn an application of the Nowak and Raghavan [40] existence theorem for correlated Markov perfect equilibrium.

Embedded into the requirement that players only employ pure Markov strategies is the behavioral assumption that players' beliefs do not change during the course of the game. That is, every time they observe the same event, their beliefs about the state of the world and other players’ types are the same. Still, even if beliefs are fixed, players may learn about the other players’ types and the state of the world if, for example, there are some states in an observed event that are not consequent of any state in the previous period’s observed event. We show that if both the player’s belief and the Markov law of motion have full support, players are not able to rule out any state from being considered possible and their beliefs do not change during
the course of the game.

In the second part of this chapter we also show that under some additional conditions on the player’s belief and the Markov law of motion, as the game becomes large, there exists a finite threshold on the number of players such that beyond this number no player is able to learn from the history of observable events during the course of the game. To establish this result we suppose the Markov law of motion on each player’s state space is decomposable between the impact of each player’s action and this impact takes the form of a product. Each player’s state space is the product of the impact of each player in the game on that player’s state space. We also require the marginal impact to satisfy two assumptions: Richness and Persistence. We can interpret the Richness assumption as saying that players are interconnected. That is, by taking the same action over and over, the probability of going from any state to every other state in a finite number of steps is positive. The Persistence assumption says that there exists a positive probability of staying in the same state in the next period if players take the same action. The last theorem in the second part of the chapter places an upper bound on the number of players needed in order to ensure that there is no learning.

We also offer some applications of our framework. The first is an imperfect competition model where there exists some privately observed variable that is is serially correlated over time. Our second application is a dynamic search game based on Diamond [12] with hidden search productivity. The last application is an arms race where the countries’ arms stock is private information.

This paper is directly related to the literature on stochastic incomplete information games and the existence of equilibrium for stochastic games. The stochastic incomplete information literature studies the class of games where there exists a
payoff relevant state variable that is not commonly known by all players and its transition is serially correlated over time. To the best of our knowledge, the only papers in this literature are Pakes and Fershtman [42], Cole and KocheKonlakota [10] and Athey and Bagwell [4]. In Cole and KocheKonlakota [10], beliefs are Markovian, e.g. are given by the observed state by all players (public state space). The authors show that if an equilibrium exists for their class of games it can be calculated using an interactive procedure based on Abreu, Pearce and Stacchetti [1] [2]. Their framework is neither contained nor contains ours. They are more general in the extent that beliefs vary over time, because the beliefs are updated based on a public observed state. However, they are a subcase of our framework in some other dimensions. For example, when we allow a richer set of information structures, beliefs may not come from a common prior, and more importantly, we are able to show equilibrium existence. On the other hand, Athey and Bagwell [4]’s objective is the study of cartel formation where the cost variable is private information and persistent over time. This paper builds upon the iterated operator technique on Cole and KocheKonlakota [10]. The article by Pakes and Fershtman [42] proposes a class of models and an equilibrium concept called applied Markov equilibrium to the study of imperfect competition games, based on Pakes and Ericson [41] (see also Doraszelski and Satterthwaite [14] for existence of equilibrium in pure strategies for this class of models) with incomplete information. In their equilibrium concept the players’ beliefs are in equilibrium. They suppose players form beliefs that are consistent with the ergodic process on the state space implied by the equilibrium strategies. Further, they do not show equilibrium existence.

The issue of equilibrium existence is important particularly from an applied point of view. The existence of equilibrium for some classes of games implies that
we are able to make predictions about that strategic situation. The literature on existence of equilibrium for complete information stochastic games is extensive (see for example Mertens and Parthasarathy [34], [40] to cite a few). The authors focus on restrictions on the transition function, such as, absolute continuity in order to guarantee existence of Markov perfect equilibrium or correlated Markov perfect equilibrium. Our paper is closely related to the literature on equilibrium existence for supermodular stochastic games (Nowak [39], Curtat [11], Amir [3], etc...), being more closely related to Nowak [39]. Still, the complete information version of the class of games studied in this paper is a subcase of Nowak [39].

The paper is divided as follows. In the next section we define the class of dynamic games of incomplete information and Markov law of motion. Then, we provide additional assumptions under which we can prove the existence of equilibrium. In the fourth section we provide sufficient conditions in which the equilibrium for the game is in Markov strategies. Following this there is a section that describes the application and then the last section concludes the paper.

1.2 Dynamic Game of Incomplete Information and Markov Law of Motion

Framework

Let $N$ be a finite set of players, $i = 1, \cdots, n$, who interact over an infinite horizon and $S$ a finite state space that summarizes all payoff relevant information about the game.

Players may not be perfectly informed about the state. We suppose that at each period $\tau$ they observe an event $B_i \subset S$ which contains the true state of the world $s_\tau$. In this paper we call $B_i$ an observable event or relevant hypothesis. We denote
by $B_i$ the set of observable events player $i$ may observe and $B_i(s_\tau)$ the event player $i$ observes whenever the true state of the world is $s_\tau$. The set of observable events of all players $\mathcal{B} = \prod B_i$ and $B(s) = (B_1(s_\tau), \ldots, B_n(s_\tau))$ is a list with the observable state each player observes whenever the state is $s_\tau$.

By choosing different $B_i$ we can vary each player’s information structure. For instance, consider the following state space $S = \prod \Omega_i \times Z$ where $w_i \in \Omega_i$ represents player $i$’s state and $z$ the aggregate state. A state of the world is a pair $(w, z) \in S$. If $B_i$ is such that $B_i(w, z) = \{w_i\} \times \Omega_{-i} \times \{z\}$, we have a private information setup where player $i$ is only able to observe his private state and the aggregate state. On the other hand, if $B_i$ is such that $B_i(w, z) = \Omega \times \{z\}$, player $i$ only observes the common state $z$. We can also handle the complete information case by letting $B_i$ be such that $B_i(w, z) = \{w\} \times \{z\}$.

As the state of the world might only be partially observed depending on the choice of $B_i$, players face uncertainty about what the other player observes, what they believe the other players observe and so on. We model the iterative uncertainty about $S$ faced by the set of players using a dynamic Harsanyi type space\footnote{See Appendix B for details about this type space.} proposed by Battigalli and Siniscalchi [5]. A dynamic type space is a tuple:

$$(N, S, (\pi_i, T_i, B_i)_{i \in N}).$$

The set $T_i$ is a finite type space. It contains the set of possible types for player $i$. For each $t_i \in T_i$ the belief mapping $\pi_i(t_i) \in \Delta^{B_i \times T_{-i}}(S \times T_{-i})$ associates a Conditional Probability System over the state space and other player’s types for type $t_i$. To simplify notation we omit $T_{-i}$ in the belief mapping. That is, instead of writing $\pi_i(t_i)[\cdot|B_i \times T_{-i}]$, we write $\pi_i(t_i)[\cdot|B_i]$. A Conditional Probability System is a list of probability distributions over the set of states and other player’s types,
one for each observable event \( B_i \in B \) that satisfies a number of properties. First, \( \pi_i(t_i)[B_i \times T_{-i}|B_i] = 1 \), \( \pi_i(t_i)[|B_i] \in \Delta(S \times T_{-i}) \) and for all \( C \in S \times T_{-i} \) and \( D, E \in B \) if \( C \subset D \times T_{-i} \subset E \times T_{-i} \) we have \( \pi_i(t_i)[C|D \times T_{-i}] \pi_i(t_i)[D \times T_{-i}|E \times T_{-i}] = \pi_i(t_i)[C|E \times T_{-i}] \). The first property says that the Conditional Probability System indexed by \( B_i \) assigns probability 1 to the event \( B_i \times T_{-i} \). The second says that it is a probability measure over \( S \times T_{-i} \) and the third that it satisfies the Bayes’ rule whenever possible. The main advantage of working with conditional probability systems is the fact that beliefs are well defined even if an observable event \( B_i \) has probability zero ex-ante\(^2\).

At each period \( \tau \) each player has a set of actions \( A_i \) available. We suppose \( A_i \) is a closed and bounded interval of the real line. We assume without loss of generality that \( A_i = A_j \) for each \( i, j \in N \). In addition, the space of actions of all players is \( A = \prod A_i \). In our model, actions are not observable. We make this assumption because we are not interested in Folk theorems, e.g. equilibrium situations that may arise depending on punishment schemes.

Player \( i \)'s payoff function is a mapping \( u_i : T_i \times S \times A \to \mathbb{R} \) where \( |u_i(t_i, s, a)| \leq C \) and \( C \) is a constant. Also, players discount future payoff at a common rate \( \delta \in [0, 1) \).

**Definition 1** A mapping \( Q : S \times A \to \Delta(S) \) is a Markov law of motion.

The Markov law of motion \( Q \) models how the state transition depends on the actual state and on the players’ actions. We call the law of motion \( Q \) Markov, because it depends only on the current state of the world, not on larger histories of states. We suppose \( Q \) is dominated by some probability measure \( \mu \) on \( S \). The support of \( Q \) is \( \text{supp} \ Q = \bigcup_{(s,a) \in S \times A} \{ s' \in S | Q(s'|a, s) > 0 \} \).

\(^2\)See Appendix B for details on Conditional Probability Systems
The timing of the game is as follows: it starts at some state $s_0 \in S$, then, players observe $B_i(s_0)$ and pick an action $a_{i,1} \in A_i$, payoff is realized and the next state of the world $s_1$ is drawn from $Q(\cdot | a_1, s_0)$ and so on.

**Definition 2** A dynamic game of incomplete information and Markov law of motion is a tuple:

$$(N, S, (\pi_i, T_i, B_i, A_i, u_i)_{i \in N}, Q, \delta).$$

**Equilibrium Existence**

In this section we define strategies, expected payoff, the equilibrium concept, and show equilibrium existence for the class of dynamic games with incomplete information and Markov law of motion.

**Definition 3** A pure Markov strategy is a mapping $\sigma_i : T_i \times B_i \to A_i$.

A pure strategy is a contingent plan that assigns an action for each possible player’s type and observable event. We call this type of pure strategy Markov because it depends only on the observable event at each period, not on any history of observable events. The space of pure Markov strategies for player $i$ is $\Sigma_i$. In addition we denote by $\Sigma = \prod \Sigma_i$ the space of pure Markov strategies of all players. As usual, when we add the subscript $-i$ we refer to all players except $i$.

From the applied standpoint it is easier to work with pure instead of behavioral strategies. If we were to simulate a model in which equilibrium might exist in behavioral strategies, it would significantly increase the computational burden. Still, when we allow the possibility of players using Markov behavioral strategies, we are able to show equilibrium for a more general setup than with pure Markov strategies. We define the notion of equilibrium in Markov behavioral strategies and prove equilibrium existence in Appendix A.
Let \( v_i : T_i \times B_i \rightarrow \mathbb{R} \) be a mapping such that \( |v_i(t_i, B_i)| \leq C \) for each \( t_i \in T_i \) and \( B_i \in B_i \). Note that \( C \) is the same constant bounding the payoff function. We call \( v_i \) a value function. It is the discounted expected future payoff for player \( i \) with type \( t_i \) starting at each observable event \( B_i \). The space of all value functions for player \( i \) is \( \mathcal{V}_i \). The space of all value functions is \( \mathcal{V} = \prod \mathcal{V}_i \), endowed with the product topology.

**Definition 4** The expected payoff for player \( i \) is a mapping \( h_i : T_i \times B_i \times \Sigma \times \mathcal{V}_i \rightarrow \mathbb{R} \) as follows:

\[
h_i(t_i, B_i; \sigma, v_i) = \sum_{(s, t_{-i})} [u_i(t_i, s, \sigma(B_i(s), t_i, t_{-i}))+ \delta \sum_{s'} Q(s'|\sigma(B_i(s), t_i, t_{-i}), s)v_i(t_i, B_i(s'))] \pi_i(t_i)[(s, t_{-i})|B_i]
\]

The function \( h_i \) is the expected payoff for type \( t_i \) upon observing \( B_i \), if his value function is \( v_i \), and players follow the pure Markov strategy \( \sigma \). The expression for the expected payoff takes into consideration the fact that type \( t_i \) may not know the actual state of the world and the other players’ types. Note that the strategy influences the likelihood of future states as it enters into the Markov law of motion.

**Definition 5** Let \( (N, S, (\pi_i, T_i, B_i, A_i, u_i)_{i \in N}, Q, \delta) \) be a dynamic incomplete of information game and Markov law of motion. A pair \( (\sigma^*, v^*) \in \Sigma \times \mathcal{V} \) is an equilibrium in pure Markov strategies if:

\[
\sigma_i(t_i, B_i) \in \arg \max_{\sigma_i \in \Sigma_i} h_i(t_i, B_i; \sigma^*_i, v_i^*) \quad (1.1)
\]

\[
v_i^*(t_i, B_i) = h_i(t_i, B_i, \sigma^*, v_i^*) \quad (1.2)
\]

for each \( B_i \in B_i \), \( t_i \in T_i \) and \( i \in N \).
Equilibrium in pure Markov strategies is a pair of two functions. The equilibrium strategy and the value function equals the expected payoff calculated using the equilibrium strategy at each observable state for each player’s type.

In the remainder of this section we present the assumptions required to guarantee existence of equilibrium in pure Markov strategies. There are restrictions in the payoff function, type space and the Markov law of motion. We start with restrictions in the payoff function.

**Assumption 1** The payoff function $u_i$ is twice continuously differentiable with respect to $a$, concave in $a_i$ and has increasing differences in $(a_i, a_j)$ for each $j \neq i$, $s \in S$, $t_i \in T_i$, e.g. $\frac{\partial^2 u_i}{\partial a_i \partial a_j}(t_i, s, a) \geq 0$.

**Assumption 2** The payoff function $u_i$ satisfies the strict diagonal dominance condition if:

$$\left| \frac{\partial^2 u_i}{\partial a_i^2}(t_i, s, a) \right| > \sum_{j \neq i} \left| \frac{\partial^2 u_i}{\partial a_i \partial a_j}(t_i, s, a) \right|$$

for each $(t_i, s, a) \in T_i \times S \times A$.

The first assumption, in addition to concavity and differentiability, requires complementarity between players’ actions. It says that when player $i$ increases his strategy, and this notion is well defined since the action set is an interval of the real line, a further increase for some player $j$ has a positive impact on player $i$’s payoff. The second assumption is also known in the literature as ”dominant diagonal condition” (Milgrom and Roberts [37], Curtat [11] ). This condition can be interpreted as follows: the player’s own payoff is affected more by their own action than by the impact of all the other players’ action together. Its main role in the existence proof is to guarantee uniqueness of equilibrium in the stage game.
**Assumption 3** Player’s beliefs satisfy the no irrelevant type property

\[ \bigcup_{t_i \in T_i} \text{supp } mrg_{T\ldots\pi_i(t_i)}[\cdot|B_i] = T_{-i} \]

for each \( B \in B_i \) and \( i \in N \).

Assumption 3 on players’ beliefs is a regularity assumption. It says that there is no type profile of other players that is considered impossible by all types of that player. That is, there is no type of some player that gets zero probability for all possible types of that player at each observable event. With this assumption we eliminate strategically irrelevant types.

**Assumption 4** Let \( \alpha_i : S \times A_i \to [0,1] \) be a linear function of \( a_i \) one for each \( i \in N \). In addition, let \( \mu_1, \mu_2 : S \to \Delta(S) \). The Markov law of motion is as follows:

\[
Q(s'|a,s) = \sum_{i \in N} \alpha_i(s,a_i)\mu_1(s'|s) + \left(1 - \sum_{i \in N} \frac{\alpha_i(s,a_i)}{n}\right)\mu_2(s'|s).
\]

Under Assumption 4 the Markov law of motion is a convex combination between two Markov chains in the state space. Depending on the choice of \( \alpha_i \), we can have some interesting interpretations of the Markov law of motion. Suppose, for example, that \( \alpha_i \) is monotone increasing in \( a_i \) for each player \( i \), \( \mu_1 \) first order stochastically dominates \( \mu_2 \) and the payoff function is increasing in \( s \). In this setup, by picking a higher action it becomes more likely that the next state of the world will be drawn from the probability distribution \( \mu_1(\cdot|s) \), which is better for player \( i \).

Whenever Assumptions 1,2 and 4 hold we can write the expected payoff function as:

\[
h_i(t_i, B_i; \sigma, v_i) = \sum_{(s,t_{-i})} \tilde{u}_i(t_i, s, \sigma(t, B(s)); v_i)\pi_i(t_i)[(s,t_{-i})|B_i].
\]

Where \( \tilde{u}_i(t_i, s, a; v_i) \) is defined as follows:
\[ \tilde{u}_i(t_i, s, a; v_i) = u_i(t_i, s, a) + \delta \sum_{s'} v_i(t_i, B_i(s')) \mu_2(s'|s) + \delta \sum_{i \in N} \alpha_i(s, a_i) D(s, t_i, v_i) \]  \hspace{1cm} (1.3)

and

\[ D(s, t_i, v_i) = \sum_{s'} v_i(t_i, B_i(s')) \mu_1(s'|s) - \sum_{s'} v_i(t_i, B_i(s')) \mu_2(s'|s). \]  \hspace{1cm} (1.4)

This way of writing the expected value function is useful in the existence of equilibrium in the stage game.

**Lemma 1** Suppose \( u_i \) satisfies Assumptions 1-2 and the Markov transition \( Q \) satisfies Assumption 4. Then, \( \tilde{u}_i \) also satisfies Assumptions 1 and 2.

**Proof.** The fact that \( \tilde{u}_i \) is twice continuously differentiable in \( a \) follows from the fact that \( u_i \) is twice continuously differentiable with respect to \( a \) and \( \frac{\partial^2 Q(a)|s}{\partial a_i \partial a_j} = 0 \) for each \( i, j \in N \).

The next step is to show that \( \tilde{u}_i \) is concave. Let \( k \in [0,1] \) and \( a_i, a'_i \in A_i \). The function \( \tilde{u}_i \) is concave if:

\[ \tilde{u}_i(t_i, s, ka_i + (1-k)a'_i; v_i) \geq k \tilde{u}_i(t_i, s, a_i; v_i) + (1-k) \tilde{u}_i(t_i, s, a'_i; v_i). \]  \hspace{1cm} (1.5)

Using the definition of \( \tilde{u}_i \):

\[ u_i(t_i, s, ka_i + (1-k)a'_i; v_i) + k \alpha_i(s, ka_i + (1-k)a_i) D(s, t_i, v_i) \geq \]

\[ \geq ku_i(t_i, s, a_i; v_i) + k \alpha_i(s, a_i) D(s, t_i, v_i) + (1-k)u_i(t_i, s, a'_i; v_i) + (1-k) \delta \alpha_i(s, a'_i) D(s, t_i, v_i), \]

where all the terms that do not depend on \( a_i \) cancel. As \( \alpha_i \) is a linear function of \( a_i \) the last equation becomes:

\[ u_i(t_i, s, ka_i + (1-k)a'_i; v_i) \geq ku_i(t_i, s, a_i; v_i) + (1-k)u_i(t_i, s, a'_i; v_i) \]  \hspace{1cm} (1.6)
which is true since $u_i$ is concave.

It remains to show that $\tilde{u}_i$ has increasing differences and satisfies the strict diagonal dominance condition. We show increasing differences first. Taking the first order derivative of $\tilde{u}_i$ with respect to $a_i$:

$$\frac{\partial \tilde{u}_i}{\partial a_i}(t_i, s, a_i) = \frac{\partial u_i}{\partial a_i}(t_i, s, a_i) + \alpha'_i(s, a_i)D(s, t_i, v_i).$$  \hfill (1.7)

Taking the second order derivative with respect to $a_j$ we get:

$$\frac{\partial^2 \tilde{u}_i}{\partial a_i \partial a_j}(t_i, s, a_i) = \frac{\partial^2 u_i}{\partial a_i \partial a_j}(t_i, s, a_i) \geq 0. \quad \hfill (1.8)$$

The last step of the proof is to show that $\tilde{u}_i$ satisfies strict diagonal dominance. Taking the second order derivative with respect to $a_i$:

$$\frac{\partial^2 \tilde{u}_i}{\partial a_i^2}(t_i, s, a_i) = \frac{\partial^2 u_i}{\partial a_i^2}(t_i, s, a_i), \quad \hfill (1.9)$$

Since the second order derivative of $\alpha'_i(s, a_i)D(s, t_i, v_i)$ with respect to $a_i$ is zero, and the payoff function has increasing differences, therefore,

$$\left| \frac{\partial^2 \tilde{u}_i}{\partial a_i^2}(t_i, s, a_i) \right| = \left| \frac{\partial^2 u_i}{\partial a_i^2}(t_i, s, a_i) \right| > \sum_{j \neq i} \left| \frac{\partial^2 u_i}{\partial a_i \partial a_j}(t_i, s, a_i) \right| = \sum_{j \neq i} \left| \frac{\partial^2 \tilde{u}_i}{\partial a_i \partial a_j}(t_i, s, a_i) \right|, \quad \hfill (1.10)$$

as we wanted to show. \hfill \blacksquare

If we fix the value function $v \in V$ the stage game at each state $s \in S$ is a static incomplete information game. In this incomplete information game beliefs are given by $\pi_i(t_i)[\cdot|B_i(s)]$ and the payoff function is $\tilde{u}_i(t_i, \cdot, B_i(s); v_i)$ for each player $i$. We denote the incomplete information game induced by $v$ at $s$ as $\Gamma(s, v)$. The first step of the proof is to show that whenever the dynamic incomplete information
game with Markov law of motion satisfies Assumptions 1-4, $\Gamma(v, s)$ has an unique Bayesian Nash equilibrium.

**Proposition 2** Suppose the dynamic incomplete information game with Markov law of motion satisfies Assumptions 1-4. Then, the induced incomplete information game $\Gamma(s, v)$ by $v \in V$ at each $s \in S$ has a unique Bayesian Nash equilibrium.

**Proof.** It follows from Lemma 2.1 that $\tilde{u}_i$ is continuous in $a$ and has increasing differences with respect to $(a_i, a_{-i})$. Then, we can apply the existence theorem in Van Zandt [51]\(^3\) to show that $\Gamma(s, v)$ has a Bayesian Nash equilibrium.

Also, it follows from Lemma 3.1 that $\tilde{u}_i$ satisfies Assumption 4. Hence, we can apply Polydoro [44] to show that the Bayesian Nash equilibrium for $\Gamma(s, v)$ is unique.

The fact that $\Gamma(s, v)$ has a unique Bayesian Nash equilibrium is an important intermediate step in the proof of equilibrium existence in pure Markov strategies. Before we proceed to the main theorem of this section we need a few additional notations. Let $BNE : S \times V \to \Sigma$ be the Bayesian Nash equilibrium correspondence for $\Gamma(s, v)$. The set $BNE(s, v)$ is composed by Bayesian Nash equilibrium points for the static incomplete information game $\Gamma(s, v)$. Define the correspondence $\Phi : V \to V$ as $\Phi(v) = \{h(t, B(s); v, \sigma) | \sigma \in BNE(s, v), s \in S\}$. That is, $\Phi(v)$ is the set of value functions consistent with the expected payoff in equilibrium for each player. In addition, let $\|\Phi(v)\| = \max_{i \in N} \max_{(t, s) \in T_i \times S} |v_i(t, B_i(s))|$. Note that a fixed point for $\Phi$ and a supporting strategy profile is equilibrium in pure Markov strategies for the dynamic incomplete information game with Markov law of motion.

\(^3\)The main theorem in Van Zandt [51] also requires the payoff function to be supermodular in $a_i$. Still, this requirement is trivially satisfied since every one dimensional function is supermodular.
Theorem 3 Suppose the dynamic game of incomplete information and Markov law of motion \((N, S, (\pi_i, T_i, B_i, A_i, u_i)_{i \in N}, Q, \delta)\) satisfies Assumptions 1-4. There exists equilibrium in pure Markov strategies.

Proof. Since the game satisfies Assumptions 1-4 it follows from Proposition 2.1 that the set of Bayesian Nash equilibrium for \(\Gamma(s, v)\) induced by a value function \(v\) at each \(s\) is a singleton. Hence \(\Phi\) is well defined function.

We show the existence of equilibrium for the dynamic incomplete information game in pure Markov strategies by proving the existence of a fixed point for \(\Phi\). We do so as an application of Brouwer’s fixed point theorem.

In order to apply Brouwer’s fixed point theorem we need to show that \(\Phi\) is continuous. The space \(\mathcal{V}\) is compact and metrizable by definition as both the set of states and types are finite.

Our first step to show continuity of \(\Phi\) is to show that for each \(s \in S\) the Bayesian game \(\Gamma(s, v^n)\) converges to \(\Gamma(s, v)\) in the sense that

\[
\lim_{n \to \infty} I_i^n(t_i, s) = \max_{\sigma \in \Sigma} |h_i(t_i, B_i(s); v^n_i; \sigma) - h_i(t_i, B_i(s); v_i; \sigma)| = 0
\]

for each \(t_i \in T_i, i \in N\), where \(\{v^n\} \to v\) pointwise.

Suppose by way of contradiction that there exists a type \(t_i\) and state \(s\) such that \(\lim_{n \to \infty} I_i^n(t_i, s) > 0\). Then, there exists a number \(\alpha > 0\) and an infinite set of integers \(J\) such that \(I_i^n(t_i, s) > \alpha\) for each \(n \in J\). Let \(\sigma^0 \in A_i \times \Sigma_{-i}\) be a strategy in which \(\alpha\) is attained for \(v^n_i\). Since \(A_i \times \Sigma_{-i}\) is compact we suppose without loss of generality that \(\sigma^n \to \sigma^0\) pointwise as \(n \to \infty\) (or else we can go to a subsequence in order to get convergence).
Then, from the definition of \( l^n_i(t_i, s) \), we have

\[
l^n_i(t_i, s) \leq \delta \left| \sum \pi_i(t_i)[(s, t_{-i})|B_i]Q(s'|\sigma^0(t, B(s)), s) (v^n_i(t_i, B_i(s')) - v_i(t_i, B_i(s'))) \right| +
+ \left| \sum \pi_i(t_i)[(s, t_{-i})|B_i] (Q(s'|\sigma^n(t, B(s)), s)v^n_i(t_i, B_i(s')) - Q(s'|\sigma^0(t, B(s)), s)v^n_i(t_i, B_i(s'))) \right| +
+ \left| \sum \pi_i(t_i)[(s, t_{-i})|B_i] (Q(s'|\sigma^n(t, B(s)), s) - Q(s'|\sigma^0(t, B(s)), s)) v_i(t_i, B_i(s')) \right|
\]

where the summation is over \((s, s', t_{-i}) \in S \times S \times \text{supp} \pi_i(t_i)[|B_i]\).

The first term converges to zero because \( Q(\cdot|\sigma^0(t, B(s)), s) < < \mu \) and \( v^n_i \to v_i \) pointwise. The other two terms are less than or equal to

\[
\max_{t_{-i} \in \text{supp } \text{mrgr}_{-i}\pi_i(t_i)[|B_i(s)]} C \left\| Q(\cdot|\sigma^n(t, B(s)), s) - Q(s'|\sigma^0(t, B(s)), s) \right\|,
\]

which converges to zero because \( Q(\cdot|\sigma^n(t, B(s)), s) \to Q(\cdot|\sigma^0(t, B(s)), s) \) using continuity of \( Q \) with respect to the action profile. Therefore we have a contradiction.

Let \( \sigma^*(s, v) = BNE(s, v) \) and define

\[
\tilde{l}^n_i(t_i, s) = \left| h_i(t_i, B_i(s); v^n_i; \sigma^*(s, v^n)) - h_i(t_i, B_i(s); v_i; \sigma^*(s, v)) \right|.
\] (1.12)

The value of \( \tilde{l}^n_i(t_i, s) \) is the difference between the expected payoff in equilibrium for the stage game induced by \( v^n \) and \( v \) at \( s \in S \).

By definition we have \( \tilde{l}^n_i(t_i, s) \leq l^n_i(t_i, s) \) for each \( t_i \in T_i, i \in N \) and \( s \in S \). Since it holds for every \((t_i, s)\) and player \( i \) we have

\[
\left\| \Phi(v^n) - \Phi(v) \right\| = \max_{i \in N} \max_{s} \max_{(s, t_i) \in S \times T_i} \tilde{l}^n_i(t_i, s) \leq \max_{i \in N} \max_{s} \max_{(s, t_i) \in S \times T_i} l^n_i(t_i, s).
\]

and since \( l^n_i \to 0 \) we have \( \lim_{n \to \infty} \left\| \Phi(v^n) - \Phi(v) \right\| = 0 \). Then, we can apply Brouwer’s fixed point theorem to show that there exists a fixed point for \( \Phi \).

**Corollary 4** Let \((N, S, (\pi_i, T_i, B_i, A_i, u_i)_{i \in N}, Q, \delta)\) be a dynamic incomplete of information game and Markov law of motion satisfying Assumption 3 such that \( \bar{u}_i \) satisfies Assumptions 1 and 2 for each \( i \in N \). Then, there exists equilibrium in pure Markov strategies.
Proof. Since the game satisfies Assumption 3 and the payoff function $\tilde{u}_i$ satisfies Assumptions 1 and 2 for each $t_i \in T_i$, $s \in S$ and $a \in A$ for each $i \in N$, it follows from Polydoro [44] that $\Gamma(s, v)$ has a unique Bayesian Nash equilibrium. Then, we can follow the same steps as in the proof of the existence theorem to establish the result.

1.3 Sufficient Conditions for equilibrium in Markov Strategies

In the previous section we showed that under some restrictions on beliefs, payoff, and the Markov law of motion, there exists equilibrium in pure Markov strategies for the class of dynamic incomplete information games with Markov law of motion.

Embedded into the Assumption that players employ pure Markov strategies, is the behavioral assumption that beliefs about the state of the world and other players’ types do not change. That is, every time type $t_i$ observes the event $B_i$, his belief is $\pi_i(t_i)\cdot |B_i]$. This is a reasonable assumption if during the course of play, players are not able to make better predictions about the state and other players’ types. They do not learn. If this is not the case, players learn during the course of the game, strategies should depend on more than the observable event and the player’s type.

By assumption, the state is a payoff relevant variable and the type of other players an informationally relevant variable. Where according to Maskin and Tirole [30] and Pakes and Fershtman [42], a variable is payoff relevant if it affects payoff and is not a current control. In addition, Pakes and Fershtman [42] define an informationally relevant variable as the set of variables that players can gain by conditioning on it but do not directly affect payoff.
In the remainder of this section, we define what we mean by learning in this class of games and provide sufficient conditions for no learning. We provide two sets of sufficient conditions for Markov strategies. In the first, we place restrictions on beliefs and the Markov law of motion. In the second set of assumptions, we show that as we increase the number of players in the game, there exists a finite threshold, such that above this number there is no learning. The last result places an upper bound on this threshold.

To show how players may learn about the state and other players’ types given a pure Markov strategy consider the following example. Suppose at \( \tau = 1 \) some player observes \( B_1 \) and in the next period he observes \( B_2 \). This player can learn about the true state of the world in \( \tau = 2 \) if there exists some state considered possible in \( B_2 \) that is not consequent of any state in \( B_1 \). Likewise, he may learn about the other player’s type profile if \( B_2 \) is not consequent of the actions taken by this type profile and a state considered possible in \( \tau = 1 \). To formalize the intuition provided in this example we need a few definitions.

**Definition 6** Let \( t_i \in T_i \), \( B_i \in B_i \) and \( \sigma \in \Sigma \). The probability distribution over the next state and player’s types according to type \( t_i \) is a mapping \( \delta_i(t_i, B_i, \sigma) : S \times T_{-i} \to [0, 1] \) as follows:

\[
\delta_i(t_i, B_i, \sigma)(s', t_{-i}) = \sum_s \pi_i(t_i)[(s, t_{-i}) | B_i]Q(s' | \sigma(B(s), t), s) \tag{1.13}
\]

The probability of a pair \((s', t_{-i})\) is positive if two conditions are met. First \( t_{-i} \) must be considered possible by \( t_i \). The second is that \( s' \) is in the support of \( Q(\cdot | \sigma(B(s), t), s) \).

**Definition 7** Let \( t_i \in T_i \), \( B_i \in B_i \) and \( \sigma \in \Sigma \). The set of consequent states of \( B_i \)


and possible types according to $t_i$ given $\sigma$ is:

$$S(t_i, B_i; \sigma) = \{(s', t_{-i}) \in S \times T_{-i} | \delta_i(t_i, B_i, \sigma)(s', t_{-i}) > 0\}.$$  

Suppose there exists $B_{\tau-1}, B_{\tau} \in B_i$ and $\sigma \in \Sigma$ such that $\text{supp } \pi_i(t_i)[\cdot|B_{\tau}] - S(t_i, B_{\tau-1}; \sigma) \neq \emptyset$. Then, there are some pairs $(s, t_{-i}) \in S \times T_{-i}$ that would be considered possible at $B_{\tau}$ if type $t_i$ did not take into consideration the event $B_{\tau-1}$. Hence, whenever $\text{supp } \pi_i(t_i)[\cdot|B_{\tau}] - S(t_i, B_{\tau-1}; \sigma)$ is empty, type $t_i$ is not able to learn by taking into consideration the previous event.

**Lemma 5** Let $\text{supp mrg}_{t, i} \pi_i(t_i)[\cdot|B_i] = T_{-i}$ for each $B_i \in B_i$, $t_i \in T_i$ and $i \in N$. In addition, suppose $\text{supp } Q((\cdot|a, s) = S$ for each $(a, s) \in A \times S$. Then, there is no learning in the game.

**Proof.** Let $B_i \in B_i$ and pick $(s, t_{-i}) \in S \times T_{-i}$ such that $\pi_i(t_i)((s, t_{-i})|B_i) > 0$. By hypothesis $Q(s'|\sigma(B(s), t_{-i}), s) > 0$ for each $s' \in S$. Therefore,

$$\pi_i(t_i)((s, t_{-i})|B_i)Q(s'|\sigma(B(s), t_{-i}), s) > 0$$

for each $s' \in S$ at $t_{-i}$, which in turn, by definition, implies that $\delta_i(t_i, B_i, \sigma)(s', t_{-i}) > 0$ for each $s' \in S$.

As there always exists a state in which some type profile of other players $t_{-i} \in T_i$ is considered possible, we have $S(t_i, B_i; \sigma) = S \times T_{-i}$. Moreover $S(t_i, B_i; \sigma) \supseteq B'_i \times T_{-i}$ for each $B'_i \in B_i$.  

The first result of this section shows that if we are willing to make the assumption that beliefs have full support on types for each observable event and the Markov law of motion has full support, then the Markov assumption on strategies is reasonable. Every time players observe the same event, they have the same beliefs because they are not able to learn during the course of the game.
In the remainder of this section, we show that we can guarantee that players do not learn with a weaker assumption than full support on the Markov law of motion, for a subclass of dynamic games with incomplete information, and the Markov law of motion as the number of players in the game becomes large. To prove this result we need to define the variable population analogue of our class of games. That is, how a game with a set of players $N$, relates to the game with the set of players $N'$ such that $N' \supset N$.

Let $N$ be the population of players and $P$ be a partition of $N$. The population of players $N$ face uncertainty about a finite state space $S = \prod_{P_k} S_{P_k} \times S_{N'}$. Each $S_{P_k}$ is the set of payoff relevant states for the set of players in the partition cell $P_k$. In addition, $S_{N'}$ is the set of payoff relevant states of all players. We suppose each $S_{P_k}$ is non-empty. One common example of $S$ is $S = \prod \Omega_i \times Z$, where $\Omega_i$ is player $i$’s state space and $Z$ is the aggregate state space. In this example, the partition $P$ contains only one element in each cell.

If the set of players in the game is $N \subset N'$, they face uncertainty about the state space $S_{N} = \prod_{P_k \in \{P_j \mid P_j \cap N \neq \emptyset\}} S_{P_k} \times S_{N'}$. The state space $S_{N}$ is the smallest subset of $S$ containing the payoff relevant state space of each player in $N$.

We can model the uncertainty faced by the population of players $N$ about $S$ using a type space:

$$(N, S, (\pi_i, T_i, B_i)_{i \in N}).$$

From this general type space we define the uncertainty faced by a subgroup of players $N$ as

$$(N, S_{N}, (\pi_{i,N}, T_{i,N}, B_{i,N})_{i \in N}), \hspace{1cm} (1.14)$$

where $T_N = \prod_{i \in N} T_i$, $B_{i,N} = \prod_{P_k \in \{P_j \mid P_j \cap N \neq \emptyset\}} B_{i,P_k}$ and $\pi_N = proj_{T_N \times S_N} \pi$. Since the group of players in the game is $N$, the relevant state space is $S_N$ and therefore
the observable events over it. Note that in the case where beliefs about the other player’s types are independent we have \( \text{mrg}_{T,i} \pi_{i,N} = \prod_{i \in N} \text{mrg}_{T,i} \pi_{ij} \).

**Definition 8** Let \( N \subseteq \mathcal{N} \). A mapping \( Q^N_{P_k} : S_{P_k} \times A^{\lvert N \rvert} \rightarrow \Delta(S_{P_k}) \) is a Markov law of motion.

A Markov law of motion for the variable population game \( Q_N = \{Q^N_{P_k}\} \) is a family of Markov law of motions, one for each possible group of players in the game and relevant state space, e.g. \( S_{P_k} \) in which \( P_k \cap N \neq \emptyset \).

Player \( i \)’s payoff in each period, when the group of players in the game is \( N \), is a mapping \( u^N_i : T_i \times S_{P(i)} \times A^{\lvert N \rvert} \rightarrow \mathbb{R} \), bounded by a constant \( C \), where \( P(i) \) is the element of \( P \) containing \( i \). Each player’s payoff in the variable population game is a family of payoff functions \( u^N_i = \{u^N_i\} \) one for each \( N \subset \mathcal{N} \), such that \( i \in N \), or else player \( i \) is not in the game and such payoff function is irrelevant. Also, players discount future payoff at a common rate \( \delta \in [0,1) \).

Although we define the game for the case where the set of players may vary, we are interested in the case where the set of players in the game is fixed during the course of the game. We choose the subgroup of players \( N \) in the game before the game starts and this is common knowledge. That is, there is no possibility of entry or exit.

**Definition 9** A variable population dynamic game of incomplete information and Markov law of motion is a tuple:

\[(\mathcal{N}, \mathcal{S}, T, \pi, A, (t_i, B_i, u_i)_{i \in \mathcal{N}}, Q, \delta)\].

Likewise, a fixed population dynamic game with incomplete information and Markov law of motion is a tuple \((N, S_N, T_N, \pi_N, A, (t_i, B_i, u^N_i)_{i \in N}, Q_N, \delta)\) where \( N \subset \mathcal{N} \).
In order to get the result on the possibility of learning as the number of players in the game becomes large, we need a few assumptions on the Markov law of motion.

**Assumption 5** Let $\alpha_{i,P_k} : S_{P_k} \times A \to [0,1]$ be a linear function of $a_i$ for each $i \in N$. In addition, $A_{i,P_k}^k, B_{i,P_k}^k : S_{P_k} \to \Delta(S_{P_k})$. The Markov law of motion over $S_{P_k}$ is as follows:

$$Q_{P_k}^N(s'|a,s) = \sum_i \frac{\alpha_{i,P_k}(a_i,s)}{|N|} \prod A_{i,P_k}^k(s'|s) + \left(1 - \sum_i \frac{\alpha_{i,P_k}(a_i,s)}{|N|}\right) \prod B_{i,P_k}^k(s'|s)$$

(1.15)

for each $P_k \cap N \neq \emptyset$ and $N \subset \mathcal{N}$.

Assumption 5 is the variable population analogue of Assumption 3 on the Markov law of motion with an additional requirement. The impact of each player in the game on the state transition in the state space $S_{P_k}$ comes into two ways. First, the convex combination between the two Markov chains on $S_{P_k}$ depends on the linear function $\alpha_{i,P_k}$. Also, each Markov chain in the definition of $Q_{P_k}^N$ is the product between $|N|$ Markov chains, one for each $i \in N$.

**Assumption 6 (Richness)** The Markov chain $A_{i,P_k}^k$ satisfies Richness if for each $s, s' \in S_{P_k}$ there exists a finite sequence $\{s_j\}_{j \leq K}$, where $s_1 = s$ and $s_K = s'$ such that $\prod_{j<K} A_{i,P_k}^k(s_{j+1}|s_j) > 0$.

Under the Richness assumption, all states in $S_{P_k}$ are reachable starting from any other state in a finite number of steps. That is, the Markov chain $A_{i,P_k}^k$ contains only one ergodic set and no transient states. This stochastic matrix has no transient states, because by definition we can always switch the first and last states in the definition. Hence, there is no state where once we leave it we never return. We can also use the same argument to show that there is only one ergodic set of states. There is no subset of states on $S_{P_k}$ where once we enter it we never leave.
Assumption 7 (Persistence) $A^P_k(s|s) > 0$ for each $s \in S_{P_k}$.

A Markov chain satisfies Persistence if the probability of staying in the same state next period is positive. That is, $A^P_k$ is a stochastic matrix with positive numbers in the diagonal. Note that player $i$ has no impact on the resulting Markov chain $A^P_k$ if it is a stochastic matrix with ones in the diagonal, e.g. the identity matrix.

For the next lemma we need to define the notion of a primitive matrix. A stochastic matrix $A$, over a finite state space $S$, is primitive if there exists a finite integer $k > 0$, such that $A^k$ contains only one self communicating class of states. That is, only one ergodic set of states and no transient states. Therefore, we can use Meyer [36][p.678] to show that $A^P_k$ is primitive. ■

We must point out that the product of two primitive matrices need not be primitive. Let $R(A)$ be the incidence matrix of a nonnegative square matrix $A$. We obtain the incidence matrix of $A$ by replacing the positive entries of $A$ with 1. For instance, take two matrices $A$ and $B$ with incidence matrices $R(A) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $R(B) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. For example, $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$. Both matrices contain only one self communicating class of states and have a positive entry in the diagonal, hence are primitive using Meyer [36][p.678]. In fact, $A^2 = \begin{bmatrix} 3/4 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ and $B^2 = \begin{bmatrix} 1/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$. Still, $R(A)R(B) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $[R(A)R(B)]^2 = R(A)R(B)$. The product of any two matrices with such incidence matrix is not a primitive matrix.
The element in the second row and first column will never be positive. We can use
the same example to show that the product of two matrices satisfying the Richness
assumption may not satisfy Richness.

**Lemma 7** Let \( A^P_i \) and \( A^P_j \) satisfy Richness and Persistence. Then both \( A^P_i A^P_j \)
and \( A^P_j A^P_i \) satisfy Richness and Persistence.

**Proof.** Fix \( s \in S_{P_k} \). We divide the proof into two steps. In the first, we show that it
satisfies Persistence and in the second, Richness. Let \( p_{ij} \) be the entry in the \( i \)th row
and \( j \)th column of \( A^P_i \), \( q_{ij} \) from \( A^P_j \) and \( c_{ij} \) of the matrix obtained by multiplying
\( A^P_i \) by \( A^P_j \) or vice-versa.

The product matrix satisfies Persistence if each entry in the diagonal is positive.
By definition \( c_{kk} = \sum_i p_{ki}q_{ik} \). In the \( k \)th position we have \( c_{kk} = p_{kk}q_{kk} + \sum_{i \neq k} p_{ki}q_{ik} \).
It follows from Persistence on \( A^P_i \) and \( A^P_j \) that \( p_{kk}q_{kk} > 0 \). Also, we are multiplying
two stochastic matrices with entries between 0 and 1, therefore \( \sum_{i \neq k} p_{ki}q_{ik} \geq 0 \).

Next, we show that the product matrix satisfies Richness. First we show that
multiplying \( A^P_i \) by \( A^P_j \) can only add more nonzero elements to \( A^P_j \) and vice-versa.
Suppose the entry \( q_{ij} \) is positive. In the same position of the product matrix we
have \( c_{ij} = \sum_k p_{ik}q_{kj} \). Still, \( p_{ii}q_{ij} > 0 \), because \( p_{ii} > 0 \) using Persistence and \( q_{ij} > 0 \)
by hypothesis. Therefore, \( c_{ij} > 0 \).

Given that the product matrix has at least the same number of positive entries of
the matrix being multiplied, \( A^P_j \) for example, for each sequence of states \( \{s_t\}_{1 \leq t \leq L} \)
on \( S_{P_k} \), such that \( s_1 = s, s_L = s' \) and \( \prod_{t \leq L} A^P_j [s_t + 1 | s_t] > 0 \), we have:

\[
\prod_{l < K} A^P_i A^P_j [s_{l+1} | s_l] > 0.
\]
each \( s, s' \in S_{P_k} \). Hence the product matrix satisfy Richness. ■
The Lemma 3.2 is an important step in the proof of the main theorem of this section.

**Theorem 8** Let $(N, S, T, \pi, A, (t_i, B_i, u_i)_{i \in N}, Q, \delta)$ be a variable population game of incomplete information and the Markov law of motion satisfying Assumption 5 and $\text{supp} \ mrg_{T_i, \pi}(t_i)[|B_i, P_k] = T_{-i}$ for each $B_i, P_k \in B_i, P_k$ and $P_k \in P$. In addition, suppose $A^{P_k}$ and $B^{P_k}$ satisfy Richness and Persistence for each $i \in N$, $P_k \cap N \neq \emptyset$ and $N \subset N$.

Then, there exists a finite $n^*(P_k)$, such that $S^N_{P_k}(t_i, B_i, P_k; \sigma) \supseteq B^{i, P_k} \times T_{-i}$ for each $B_i, P_k, B^{i, P_k} \in B_i, P_k$, $P_k \in P$ with $P_k \cap N \neq \emptyset$, $N \subset N$ in which $|N| \geq n^*$ and $\sigma \in \Sigma$.

**Proof.** Fix $N \subset N$ and let $P_k$ be such that $P_k \cap N \neq \emptyset$. From Lemma 3.2 $A^{N}_{P_k}$ and $B^{N}_{P_k}$ satisfy Richness and Persistence. In addition, Lemma 3.1 guarantees that both $A^{N}_{P_k}$ and $B^{N}_{P_k}$ are primitive matrices. It follows from Seneta [47][Lemma 3.9 p.97] that there exists a finite number $n^*_A(P_k)$ such that $A^{N}_{P_k}[s'|s] > 0$ for each $s, s' \in S_{P_k}$ and $N \in N$ such that $|N| \geq n^*(P_k)$. We can apply the same argument to $B^{N}_{P_k}$ to show that there exists a finite $n^*_B(P_k)$, such that this is a primitive matrix. As $Q^{N}_{P_k}$ is the convex combination between $A^{N}_{P_k}$ and $B^{N}_{P_k}$ when the number of players reaches $n^*(P_k) = \max\{n^*_A(P_k), n^*_B(P_k)\}$ it becomes a Primitive matrix for each $a \in A^{n^*(P_k)}$.

Now for a fixed $t_i \in T_i$, pick $B_i, P_k, B^{i, P_k} \in B_i, P_k$, $t^{N}_{-i} \in T_{-i}$ and $s \in S_{P_k}$ such that $\pi_i, N(t_i)[(s, t^{N}_{-i})|B_i, P_k] > 0$. Whenever $|N| \geq n^*(P_k)$ and $P_k \cap N \neq \emptyset$ we have $\delta_i, P_k(t_i, B_i, P_k; \sigma)(s', t^{N}_{-i}) > 0$ for each $s' \in S_{P_k}$. Therefore $S_i, P_k(t_i, B_i, P_k; \sigma) = S_{P_k} \times T^{N}_{-i} \supseteq B^{i, P_k} \times T^{N}_{-i}$. ■

This theorem shows that whenever beliefs satisfy full support on types of other players and the individual impact of each player on both the Markov chains satisfy Richness and Persistence there exists a finite number of players such that as we
increase the number of players beyond this number no one is able to learn from past observable events. That is, each player is not able to rule out any state in the actual period as not being consequent of some state in the previous observable event. The role of the Richness and Persistence in the proof is to ensure that as we add more players to the game, their impact on $S_{P_k}$ accumulates in such a way that any state is possible starting from any state in $S_{N_k}$ for each player.

Theorem 4.1 does not show what is the value of $n^*(P_k)$ only that it is finite. In the next theorem we place an upper bound on the number of players needed to have the no learning property.

**Theorem 9** Let $P_k \in P$ and $N \in \mathcal{N}$ such that $A_{i}^{P_k}$ and $B_{i}^{P_k}$ satisfy Richness and Persistence for each $i \in N$. Then, $n^*(P_k) \leq 2^{|S_{P_k}|} - 2$.

**Proof.** The set of Markov chains on $S_{P_k}$ satisfying Richness and Persistence are also Primitive (Lemma 3.2) and this set is closed under the product operation. Hence, from Cohen and Sellers [9][Theorem 1] we need at most $2^{|S_{P_k}|} - 2$ players in the game to ensure that either $A^{P_k}$ or $B^{P_k}$ have positive values in all entries. □

### 1.4 Applications

In this section we provide three applications of the framework proposed in this article and existence result. The first application is a dynamic arms race model under incomplete information based on Milgrom and Roberts [37]. The second is an imperfect market competition model under incomplete information based on Pakes and Ericson [41] and Pakes and Fershtman [42]. The third and last application is a dynamic search model based on Diamond [12].
1.4.1 Dynamic Arms Race under Incomplete Information

Suppose there are two countries engaged in an arms race for more than one period. We denote by \( y_{n,\tau} \in \{0, y_{\text{max}}\} \) country \( n \)'s arms level in period \( \tau \). There is only a finite set of possible arms levels. As countries may change their arms level over time we define the game's state space as \( S = \{0, \cdots, y_{\text{max}}\} \times \{0, \cdots, y_{\text{max}}\} \).

As the equilibrium existence theorem does not depend on the choice of \( \mathcal{B} \), we can handle different information setups about each country's arms level. For example, if \( B_n(y) = \{y_n\} \times \{0, \cdots, y_{\text{max}}\} \) for \( n = 1, 2 \) each country only observes their own arms level. On the other hand, if \( B_n(y) = \{y_n\} \times \{y_{-n}\} \) for \( n = 1, 2 \) there is complete information about each country's arms level. It could also be the case that \( B_n(y) = \{y_n\} \times \{y_{-n}\} \), but \( B_{-n}(y) = \{y_{-n}\} \times \{0, \cdots, y_n\} \). Then, one country knows the others’ arms level but the others do not.

The finite type space \( T = T_n \times T_{-n} \) summarizes each country's belief about the overall arms level and the other country's type. Each country's belief about the other country's arms level and type is a mapping \( \lambda_n : T_n \to \Delta^{B_n} \{0, \cdots, y_{\text{max}}\} \times \{0, \cdots, y_{\text{max}}\} \times T_{-i} \).

At each period each country can invest \( x_n \in [0, 1] \) in arms. A pure Markov arms investment strategy is a mapping \( \sigma : \mathcal{B} \times T_i \to [0, 1] \).

We suppose investment’s success is random. Still, by investing more, each country can guarantee a higher arms level is more likely. That is, country \( n \)'s arms level follows a Markov law of motion:

\[
Q(y_{n}', x_n, y_n) = x_n\mu_1(y_n'|y_n) + (1 - x_n)\mu_2(y_n'|y_n). \tag{1.16}
\]

Where \( \mu_1 \) first order stochastically dominates \( \mu_2 \). As \( \mu_1 \) and \( \mu_2 \) are the same for both countries we suppose that no country has comparative advantage in arms investment. We could also have the case in which \( \mu_1 \) and \( \mu_2 \) vary across countries.
Each country’s payoff is given by:

$$f_n(y_n, x_n, x_{-n}) = -C(y_n + x_n) + B(y_{-n} + x_{-n}), \quad (1.17)$$

where $C(\cdot)$ is smooth strictly concave and $B(\cdot)$ is smooth concave. In addition, whenever $C''(\cdot) > B''(\cdot)$ Assumption 1 and 2 are satisfied and we can apply Theorem 1 to show that this dynamic arms race game under incomplete information has an equilibrium in pure Markov arms investment strategies.

### 1.4.2 Imperfect Market Competition under Persistent Private Information

In this section we study a dynamic imperfect market competition model based on Pakes and Ericson [41], Pakes and Fershtman [42] and Doraszelski and Satterwaite [14].

Suppose there exists a finite number of firms $K$ in an imperfect competition products market. We endow each firm with a finite state space $\Omega_k$ that represents their characteristics or product characteristics, such as quality, durability, productivity, production costs, etc. There is also a finite aggregate state space $Z$ representing overall market conditions. The industry’s state space is $S = \prod \Omega_k \times Z$.

In the framework developed in this article we can handle different information setups about the industry state space. For example, if we set $B_k(s) = \{w_k\} \times \{z\} \times \Omega_{-k}$ firm $k$ is only able to observe their own state and the public state, but not the other firm’s state. In this case we have a private information setup. For the usual complete information setup $B_k(s) = \{w\} \times \{z\}$. We could also have different information structures, such as $B_k(s) = \{w_k\} \times \{w_{k+1}\}\{z\} \times \Omega_{-k,k+1}$ in which firm $k$ knows their own state and firm $k + 1$’s state. This type of information structure may be useful to study industry models in which firms with the same technology...
know each other’s product quality but do not know this information for other firms with different production technology.

We denote each firm’s belief type by $t_k \in T_k$, where $T_k$ is a finite type space and the belief mapping by $\lambda_k : T_k \rightarrow \Delta^B(S \times T_{-k})$.

Firms can improve their state variable through investment, which we denote by $x_k \in X_k = [0, 1]$. Each firm’s state follows a Markov law of motion:

$$Q_k(w'_k|x_k, w_k) = x_k\mu^1_k(w'_k|w_k) + (1 - x_k)\mu^2_k(w'_k|w_k). \quad (1.18)$$

The aggregate state $z$ also follows a Markov law of motion:

$$Q_z(z'|x, z) = \sum_k \alpha_k(z, x_k) \frac{\mu^1_1(z'|z)}{K} + (1 - \sum_k \alpha_k(z, x_k)) \frac{\mu^2_1(z'|z)}{K} \quad (1.19)$$

where $\alpha_k : Z \times X_k \rightarrow [0, 1]$ is a linear function on $x_k$.

A pure Markov strategy for firm $k$ is a mapping $\sigma_k : T_k \times B_k \rightarrow [0, 1]$.

We suppose as in Pakes and Ericson [41] that profits follow a static-dynamic breakdown, e.g. whatever the model for stage competition either Bertrand or Cournot, each firm’s profit can be determined by the industry’s state alone. The profit function for firm $k$ is $\pi_k : S \rightarrow [0, \pi^M]$, where $\pi^M$ is the monopolist’s profit. Embedded into this payoff formulation is the assumption that whatever prices or quantities firms choose, it does not influence state transition. This type of assumption may not be reasonable, for example in industries where there is learning by doing regarding market characteristics or production technology. Under these assumptions in the structure of the game we can apply Corollary 1 of Theorem 1 to show that equilibrium exists in pure Markov strategy.

The main contribution of the framework developed in this article on dynamic imperfect competition models is to guarantee equilibrium existence when the state variable is private information and it is serially correlated over time. This extension
is useful, because it is more likely that the firm’s characteristics, that we represent in \( w_k \) will be serially correlated such as scrap value, production costs, product quality and so on.

### 1.4.3 Dynamic Search Model with Hidden Search Productivity

In this section we study the dynamic version of a Diamond search model studied in Nowak [39], Curtat [11], among others, where search productivity may be private information.

In our version of this model there are two players who exert effort searching for trade partners. Each worker’s productivity level in the search activity is \( s_k \in S_k = \{0, \cdots, 1\} \), where \( S_k \) is a finite set for \( k = 1, 2 \). As players engage in the search activity their productivity level may vary over time. Hence, we define the game’s state space as \( S = S_1 \times S_2 \).

Player’s information about search productivity depends on the choice of \( B_k \). For the version of this model with hidden search productivity, each player’s information set is \( B_k(s) = \{s_k\} \times S_k \). For the standard complete information version, we define \( B_k(s) = \{s\} \). Moreover, since the equilibrium existence theorem does not depend on the choice of \( B \), we can handle other kinds of information structures.

Each player’s belief about the other player’s search productivity is a mapping \( \lambda_k \in \Delta^{B_k}(S_{-k}) \). In this case we don’t allow belief uncertainty. That is, if some player \( k \) knows the other player’s observed event, he also knows his belief about the productivity level. In this case there is only uncertainty about the state of the world.

At each period each player decides how much effort to exert in the search activity. We denote player \( k \)'s search level by \( x_k \in [0, 1] \). A pure Markov search strategy is a
mapping \( \sigma_k : B_i \to [0, 1] \).

The search productivity level may vary over time depending on each player’s effort level and on his search productivity in that state. For example, we can handle the case where a learning by doing effect exists on search productivity. That is, as players exert more effort on searching it becomes more likely that their productivity will be higher in the next period. For example, each player’s search productivity transition can be:

\[
Q_k(s'_k|x_k, s_k) = x_k \mu_1^k(s'_k|s_k) + (1 - x_k) \mu_2^k(s'_k|s_k),
\]  

(1.20)

where \( \mu_1^k \) first order stochastically dominates \( \mu_2^k \). Under this specification of the Markov law of motion \( Q_k \) as players search with a higher intensity, it becomes more likely that their search productivity will be higher in the next period. That is, in this model the search activity has a learning by doing effect.

Next we define the player’s payoff function as

\[
u_k(s, x) = s_k s_{-k} x_k x_{-k} - x_k^2.
\]  

(1.21)

With this payoff specification each partner’s payoff depends not only on both search effort, but also on their productivity level. As people become more productive in the search activity they contribute more to the team. Still, payoff decreases with their own effort. The payoff function is twice continuously differentiable with respect to \( x \) and clearly concave in \( x_k \). In addition \( \frac{\partial u_k}{\partial x_k}(s, x) = s_k s_{-k} x_{-k} - 2x_k \) with second order derivate with respect to \( x_k \) and \( x_{-k} \) equal to \( \frac{\partial^2 u_k}{\partial x_k^2}(s, x) = -2 \) and \( \frac{\partial^2 u_k}{\partial x_k \partial x_{-k}}(s, x) = s_k s_{-k} \). As \( s_k \in \{0, \cdots, 1\} \) for both \( k = 1, 2 \) it must be that \( \frac{\partial^2 u_k}{\partial x_k \partial x_{-k}}(s, x) = s_k s_{-k} \geq 0 \), therefore the payoff function has increasing differences. Moreover \( \frac{\partial^2 u_k}{\partial x_k^2}(s, x) = 2 > s_k s_{-k} = \left| \frac{\partial^2 u_k}{\partial x_k \partial x_{-k}}(s, x) \right| \) for each \( s_k, s_{-k} \in \{0, \cdots, 1\} \), so the payoff function also satisfies the strict diagonal dominance condition and we
can apply Theorem 1 to show that a dynamic search model with hidden search productivity has equilibrium in pure Markov search strategies.

1.5 Conclusion

We propose in this chapter a new class of dynamic stochastic games. This class of games is an extension of the standard complete information dynamic stochastic games suitable to the study of strategic situations where there is incomplete information about the state of the world.

The existence theorem holds for pure Markov strategies and depends on a few restrictions in the framework. We restrict the payoff function, the Markov law of motion, and the type space. These assumptions can be classified into two sets. The first set is needed to get the existence of a Bayesian Nash equilibrium in the static incomplete information stage game induced at each state given a value function. The second set of assumptions is needed to guarantee that the Bayesian Nash equilibrium of the stage game is unique. Our equilibrium existence theorem imposes no constraints on the information structure.

Embedded into the requirement that players employ Markov strategies, is the behavioral assumption that their beliefs do not vary over time. That is, their beliefs about the state of the world and other players’ types are completely specified ex ante. It does not depend on the path of observable that occurred during the game. We define precisely how players may learn about the state of the world and other players’ types, and show that under full support on the belief mapping and the Markov law of motion, the Markov assumption on strategies is reasonable. As the game unfolds they are not able to rule out any state or type from being considered possible.
Under additional assumptions on the Markov law of motion, we can show that as the number of players in the game becomes large, there exists a finite threshold such that above this number no player is able to learn during the game. We also characterize the upper bound of this threshold.
Chapter 2

The Role of Information in Multistage R&D Races

2.1 Introduction

R&D races are inherently dynamic processes. The innovation process usually requires the development of several intermediate stages before the product is completed. In addition, in many markets, each firm may not know the other firm’s development stage and this information is strategic. Secrecy surrounding the development of new products is an important characteristic of industry competition. One example is the computer industry.

We consider a two-firm multistage R&D race where a firm’s success in advancing from one stage of development to the other is a stochastic process that depends on the firm’s investment. As the firms invest more, it becomes more likely that they will advance to the next development stage in the next period. The cost of investing in R&D may vary across stages of development. We also allow the possibility of the firms exhibiting better abilities in R&D than their competitor’s at some stages of the race.

The objective of this paper is to extend this basic multistage R&D race to study the impact of incomplete information in relation to the other firm’s development
stage. We suppose each firm may not be perfectly informed about the other firm’s development stage. However, as the firms go along the development process, they may receive some information about the other firm’s development stage and adjust their investment accordingly.

This class of multistage R&D models is general enough to study different types of information structures. Both the complete information structure, where each firm knows exactly the other firm’s stage, and the private information structure, where the firm only knows its own stage can be studied using this class of models. In our model, the stage in the R&D race may be a privately informed variable that is persistent over time.

In the first part of the paper we define precisely the incomplete information R&D race and prove existence the of equilibrium in pure Markov investment strategies. We call the investment strategies Markov because they do not depend on the history of information regarding the other firm’s stage, only the information at that period. The existence proof is an application of Polydoro [45]. The author proves existence of equilibrium for a general class of stochastic incomplete information games, where the law of motion in the state space is Markov.

We provide a numerical analysis of these R&D races under four different information structures. For each information structure we compare the case where there are no patents and the case where there is a strong patent regime. The information structures are linearly ordered. That is, when comparing two information structures we can identify in which structure the firms have more information.

When there are no patents at the end of the race, the first firm to complete the project gets the monopolistic profits in the product’s market. Once the second firm completes the project, they share the duopolistic profits equally. In our patent
regime, the first firm to complete the project gets a patent with no expiration and
becomes the monopolist in the product’s market forever. In both cases, intermediate
developments and the whole product cannot be imitated. Firms need to develop
their own product in order to sell it on the products market.

These two types of environments are good approximations for some industries.
In the computing industry, when Apple launched the Ipad a patent was awarded to
the product, but not to the idea. Then firms started developing their own hardware
and software to make a touchscreen tablet and compete with the Ipad. In the
pharmaceutical industry, once a new drug is developed a patent is awarded and the
product cannot be imitated. Although a drug patent does not last forever we can
think of this as an example of our patents regime.

Our comparison between information structures and patent regimes focuses on
three key variables of the model. The first is the size of the product’s market, the
second is the cost of investing in R&D, and the third is the degree of competition
in the product’s market.

In our simulated model, optimal investment strategies increase as firms go along
the race. When firms are closer to the end of the race it becomes more likely that
they will finish the product and start receiving the monopolistic (or duopolistic)
profits. Further, firms increase investment as a response to the advance of the other
firm when they are leading the race and close to the end of the race. On the other
hand, firms lower investment when the other firm advances if they are behind in the
race.

For the class of R&D models we study in this paper, more information is good
for firms and consumers. Both consumer surplus and firm value is higher when firms
know more about the other firm’s development stage. With more information they
are able make a better assessment if a marginal investment in R&D will outweigh its cost. Furthermore, the race duration decreases when firms have more information resulting in higher consumer surplus. Firm value also increases. The difference in welfare between information structures increases when both market size and investment cost increase. The intensity of market competition, on the other hand, has the opposite effect. When firms are more able to coordinate their production in the products market duopoly, the difference in consumer surplus, firm value and total welfare is smaller across information structures.

The impact of adopting a strong patent regime where the first firm to complete the project gets an infinitely lasting patent is not the same on each information structure. Its impact depends on the combination between market size and development costs. Still, the effect of a patent regime is not necessarily good for consumer surplus. In general, when the market is small, a patent regime yields higher consumer surplus, but when the market is bigger, having no patents is better. Total welfare and firm value, on the other hand, are higher for the no patents regime.

This paper is directly related to Fershtman and Markovich [16]. The authors study the impact of different patent regimes on a complete information multistage R&D model where firm cost profiles may differ. That is, there is cost asymmetry between firms. The model presented in this paper is more general because it takes into consideration the possibility that firms may not be perfectly informed about the competitor’s development stage. Still, as the objective of this paper is the to study the effect of information on the race we rule out the possibility of cost asymmetries between firms.

Multistage R&D races have a long tradition in the industrial organization literature (see for example, Fudenberg et al [17], Grossman and Shapiro [20], Harris and
Vickers [22], Doraszelski [13], Judd [24] to cite a few). Unlike Fudenberg et al [17], Grossman and Shapiro [20] and Harris and Vickers [22]’s model, we don’t have an information lag, we only have information constraints. Firm’s may not be able to assess correctly in which part of the race the competitor is, but when they switch to the other part they receive this information. Also, both firms start the R&D race in the same stage, at the same time, and there is no leader or follower ex-ante. Firms do not accumulate experience: the probability of success of their investment in R&D is random and firms only advance one period at a time. Moreover, our model’s optimal strategy does not exhibit the ”leapfrogging” effect. When the firm knows that it may be behind in the race, it lowers investment.

This paper is organized as follows: in the next section we present the benchmark R&D race model and prove equilibrium existence; then, we simulate the model and analyze the impact of some key variables of the model on the race duration and welfare; and the last section concludes the paper.

2.2 Model

In this section we describe our benchmark incomplete information R&D model and prove the existence of equilibrium for this class of models.

We study a multistage R&D race model where two firms compete over the development of a new product or technology. In order to finish the product, each firm must complete \( k > 1 \) intermediate steps. We denote by \( k + 1 \) the state in which the firm has completed the product and started selling it on the market. In this benchmark case there are no patents. That is, the other firm can keep investing in the project and once it finishes both firms share the products’ market. We also consider the case where a patent with no expiration is awarded to the first firm to
complete the project.

In our R&D model, each firm is perfectly informed about its development stage, but may not know the other firm’s development stage. For example, suppose that in order to complete a project, firms need to complete four intermediate steps, e.g. \( \bar{k} = 4 \). It could be that each firm only knows whether the other is in the first half \{1, 2\} or in the second half \{3, 4\}, but not their exact stage.

Let \( K_i = \{1, \cdots, \bar{k} + 1\} \) be the set of development stages of the race for firm \( i = 1, 2 \). We model each firm’s information about the other firm’s development stage as a partition of \( K_{-i} \), which we denote by \( P^i \). We assume that the other firm always knows whether or not the firm \( i \) has completed the project. That is, \( P^i \) contains \( \bar{k} + 1 \). Then, firm \( i \)’s information set at stage \((k_i, k_{-i})\) is composed by its own stage \( k_i \in K_i \) and the element \( P_i(k_{-i}) \in P^i \) containing \( k_{-i} \). That is, firm \( i \)’s information set at \((k_i, k_{-i}) \in K_i \times K_{-i}\) is \( B_i(k_i, k_{-i}) = \{k_i\} \times P_i(k_{-i}) \). We denote the family of possible information sets for firm \( i \) by \( B_i = K_i \times P^i \).

By setting \( P_i \) accordingly, we can have different information structures about the stage of the R&D race. If \( P_i(k_{-i}) = \{k_{-i}\} \) for each \( k_{-i} \in K_{-i} \), we have a complete information setup where each firm knows the competitor’s development stage. On the other hand if \( P_i = \{1, \cdots, \bar{k}\} \) for each \( k_{-i} \in \{1, \cdots, \bar{k}\} \) we have the private information case where each firm only knows its development stage during the race. We can also have other types of information structures. For example, if \( \bar{k} \) is odd, 

\[
P_i(k_{-i}) = \{1, \frac{\bar{k}}{2}\} \text{ if } k_{-i} \leq \bar{k}, \quad P_i(k_{-i}) = \{\frac{\bar{k}}{2} + 1, \cdots, \bar{k}\} \text{ for } k > \frac{\bar{k}}{2} + 1 \text{ and } k \leq \bar{k} \quad \text{and}
\]

\[
P_i(k_{-i}) = \bar{k} + 1 \text{ if } k_{-i} = \bar{k}_{-i} + 1, \text{ firm } i \text{ knows whether the other firm is in the first half, the second half of the race, or if it has already finished. Note that } P_i \text{ may not be the same for both firms. This framework is general enough to handle the case where one firm has more information during the race than the other.}
In our setup, each firm may be uncertain about the other firm’s development stage, uncertain about what the other firm thinks is their development stage, and so on. To model this interactive uncertainty we endow each firm with beliefs about the other firm’s development stage for each information set in $B_i$. Hence, firm $i$’s belief is a list of probability distributions $\lambda_i \in \Delta_i^B(K_{-i})$, one for each information set in $B_i \in B_i$, and it does not change over time.

Firms can only advance one stage at a time. Also, moving from development stage $k_i$ to $k_i + 1$ is a stochastic process depending on the firm’s investment in R&D. Let $x_i \in [0,1]$ denote firm $i$’s investment. We assume that the probability of success at stage $k_i$ for firm $i$ is a linear combination between two probability distributions $\mu_{1,i}(\cdot|k_i), \mu_{2,i}(\cdot|k_i) \in \Delta(K_i)$:

$$Q_i(x_i)(k_i + 1|k_i) = x_i \mu_{1,i}(k_i + 1|k_i) + (1 - x_i) \mu_{2,i}(k_i + 1|k_i),$$

where $\mu_{1,i}(k_i + 1|k_i) = p_{1,i}$, $\mu_{1,i}(k_i|k_i) = 1 - p_{1,i}$, $\mu_{2,i}(k_i + 1|k_i) = p_{2,i}$ and $\mu_{2,i}(k_i + 1|k_i) = 1 - p_{2,i}$ for each $k_i < \bar{k}$. If $k_i = \bar{k} + 1$, then firm $i$ is at the end of the race and $Q_i(x_i)(\bar{k} + 1|\bar{k} + 1) = 1$ for each $x_i \in [0,1]$.

We suppose $p_{1,i} > p_{2,i}$. That is, $\mu_{1,i}$ first order stochastically dominates $\mu_{2,i}$ at each $k_i \in K_i$. With this assumption on the R&D process, as firms invest more in R&D, it becomes more likely that they will advance to the next stage. At $\bar{k} + 1$ the product or project is already completed and the firm remains in that state forever regardless of its their investment level. If firm $i$ is unsuccessful on moving from stage $k_i$ to stage $k_i + 1$, it can try again in the next period.

The cost of investing $x_i$ to advance for the $k_i + 1 - th$ development stage is $c_{i,k}x_i^2$.

---

1In fact, firm $i$’s belief is a Conditional Probability System (Myerson [38]). That is, a list of conditional probability distributions one for each $B_i \in B_i$ that is well defined even if $B_i$ has zero probability ex-ante. For details on Conditional Probabilities Systems and more specifically on the general class of games that contains the model presented in this paper see Polydoro [45].
where $c_{i,k}$ is a positive constant. The cost $c_{i,k}$ may vary across firms. That is, firms can have a comparative advantage over the other firm in developing the product at some stage. For example, in the pharmaceutical industry small firms have a comparative advantage in creating new drugs, but larger firms have an advantage once the drug reaches the FDA approval stage as this stage requires large amounts of investment in testing and paperwork filing. We denote firm $i$’s cost profile by $c_i = \{c_{i,1}, \ldots, c_{i,k}\}$.

Once the first firm reaches the $k + 1$-th stage it starts receiving the reward $\pi^M$, which is the monopolistic profit in the product’s market at each period until the other firm completes the project. When both firms reach the end of the race, the market becomes a duopoly and each firm’s profit is $\pi^D < \frac{\pi^M}{2}$. In the application section we consider the case where a patent that lasts forever is awarded to the first firm completing the project. Firms discount future profit at rate $\beta$.

**Definition 10** A multistage incomplete information R&D race $R$ is a tuple: $(N = 2, \bar{k}, (P^i, \lambda_i, c_i)_{i \in N}, \pi^M, \pi^D, \beta)$.

In this paper, a firm’s investment strategy depends only on its information set and is not randomized. That is, its choice of investment level does not depend on any history of observable events, e.g. its investment strategy is Markovian. A pure Markov investment strategy for firm $i$ is a function $\sigma_i : \mathcal{B}_i \to [0, 1]$. The space of strategies for firm $i$ is $\Sigma_i$ and the space of strategies for both firms is $\Sigma = \times \Sigma_i$.

As the R&D race may last for many periods, it is useful to present its discounted payoff in a recursive way. First, define a value function for firm $i$ as a mapping $V_i : \mathcal{B}_i \to [0, \frac{\pi^M}{1-\beta}]$ that summarizes expected discounted payoff of future periods. Given a value function and a strategy profile we can define the expected profit by a
function $h_i : B_i \times \Sigma \times V_i \to [0, \frac{\pi M}{1-\beta}]$ as follows:

$$
    h_i(B_i; \sigma, v_i) = \pi(B_i, \sigma_i) + \beta \sum_{k \in K} \sum_{k' \in K} Q(k'|\sigma(B_i, B_{-i}(k)), k) \lambda(k_{-i}|B_i)V_i(B_i(k')),
$$

(2.1)

where $\pi(B_i, \sigma_i) = A(B_i) - c_{i, k_i}\sigma_i(B_i)^2$ and $A(B_i) = \pi M$ if $k_i = \{\bar{k} + 1\}$ and $P_i \neq \{\bar{k} + 1\}$, $A(B_i) = \pi D$ if $k_i = \{\bar{k} + 1\}$ and $P_i = \{\bar{k} + 1\}$ and zero otherwise.

For the case where a patent is awarded to the first firm to complete the project we adjust this formulation accordingly.

An equilibrium in pure Markov investment strategies for a multistage incomplete information R&D race is a list of functions $(\sigma_i^*, V_i)_{i \in N}$ such that $\sigma_i^*(B_i) \in \arg \max_{x_i \in [0,1]} h_i(B_i; x_i, \sigma_i^*_{-i}, V_i)$ and $V_i(B_i) = h_i(B_i, \sigma_i^*, V_i)$ for each $B_i \in B_i$. Equilibrium for this race is such that each firm picks an investment level that maximizes expected discounted profits given that the other firm follows their equilibrium strategy for each information set. In addition, the value function is equal to the expected profit in the equilibrium.

**Proposition 10** There exists an equilibrium in pure Markov investment strategies for a dynamic R&D race with incomplete information.

**Proof.** Fix a dynamic incomplete information R&D race $R$. To show equilibrium existence in pure Markov investment strategies for this game we apply the existence theorem in Polydoro [45]. Hence, we have to show that $R$ satisfies the theorem requirements. That is, the payoff function is twice continuously differentiable with respect to the player’s actions, concave in the player’s own action and has increasing differences and satisfies the strict diagonal dominance condition (Gabay and Moulin [18]). The author’s existence theorem also imposes some restrictions on the belief mapping and on the stochastic process over the state space. Players’ beliefs are
required to have finite support and the Markov law of motion is required to be a linear combination between two probability distributions over the state space for each state.

The last two sets of assumptions, on the belief mapping and on the Markov law of motion are trivially satisfied. The belief mapping has finite support because the state space is finite, and from the definition of a dynamic R&D race with incomplete information, the Markov law of motion is a linear combination between two probability distributions.

It remains to show that the profit function satisfies the theorem’s requirements. The profit function is clearly twice continuously differentiable in \( x \) and concave in \( x_i \). Next, we show that it also satisfies increasing differences \( \frac{\partial^2 \pi_i(k, x_i)}{\partial x_i \partial x_{-i}} \geq 0 \) and strict diagonal dominance \( \left| \frac{\partial^2 \pi_i(k, x_i)}{\partial x_i \partial x_{-i}} \right| > \left| \frac{\partial^2 \pi_i(k, x_i)}{\partial x_i \partial x_{-i}} \right| \). Taking the second order derivative of the profit function with respect to investment we get \( \frac{\partial^2 \pi_i}{\partial x_i \partial x_{-i}}(k, x_i) = 0 \) and \( \left| \frac{\partial^2 \pi_i(k, x_i)}{\partial x_i^2} \right| = 2c_{i,k} > 0 \). Therefore, both conditions are satisfied.

Now we can apply the existence theorem in Polydoro [45] to show that there exists an equilibrium in pure Markov investment strategies for the multistage incomplete information R&D race \( R \).

**2.3 Analysis of R&D races**

In this section we study the impact of different information setups on a simulated multistage R&D race model. First, we present the parameters of the model. Then we present details of the numerical analysis, the interpretation of the equilibrium strategies and value function; and at the end of this section we present some comparative statistics on key parameters of the model.

---

2 The no-delusion property, which states that there is no type of other players that are considered impossible by all other players' types is trivially satisfied since there is only one belief type.
2.3.1 Parameter Values

In our simulations each period corresponds to a quarter. We set the discount rate \( \beta = 0.97 \), which corresponds to an annual interest rate of 10%. In order to capture the effect of the dynamic of the model and allow different information structures, we set the length of the race to \( \bar{k} = 6 \). That is, firms need to complete 6 intermediate steps to complete the product and start selling it.

We suppose the demand function in the product’s market is given by \( p = \frac{20}{100} - \frac{q}{100} \) and that there are no production costs. Under this specification of the demand function, monopolistic profits are \( \pi^M = 1 \) and the consumer surplus at the monopolistic price is \( CS^M = .5 \). The duopolistic payoff is \( \pi^D = \mu \pi^M \), where \( 0 \leq \mu < .5 \) captures the intensity of duopolistic competition in the product’s market. For example, if \( \mu = 0 \), we have a Bertrand competition. On the other hand, if \( \mu = .4 \), we have some type of collusive set up. The optimal quantity to be produced by both firms in the duopolistic market for a fixed \( \mu \) is given by:

\[
q^D(\mu) = 10 + 10\sqrt{1 - 2\mu}.
\]

At the optimal duopolistic quantity \( q^D(\mu) \), consumer surplus is given by \( CS^D(\mu) = \frac{q^D(\mu)^2}{200} \).

We study a cost structure where firms have different abilities to invest in R&D at different stages of the race: \( c_i = (\delta, \delta, \delta, 1, 1, 1) \) where \( \delta \geq 1 \). Under this cost structure specification firms are better at developing the product in earlier stages of the race. We suppose both firms have the same cost structure.

To capture the fact that both firms know when the race starts we add an auxiliary state that corresponds to the first development stage and denote it by 0. Then, the race state space is \( K = \{0, 1, \cdots, 6, 7\} \), where the state 7 corresponds to the end of
the race.

We consider four different information structures for the middle of the race. The first is a private information structure. Each firm only knows its development stage and does not have any information about the other firm’s development stage: \( B^1_i(k_i, k_{-i}) = \{k_i\} \times \{1, \cdots, 6\} \) for \( k_{-i} \in \{1, \cdots, 6\} \). In the second information structure, each firm knows if the other firm is in the first or the second half of the race: \( B^2_i(k_i, k_{-i}) = \{k_i\} \times \{1, 2, 3\} \) if \( k_{-i} \in \{1, 2, 3\} \) and \( B^2_i(k_i, k_{-i}) = \{k_i\} \times \{4, 5, 6\} \) if \( k_{-i} \in \{4, 5, 6\} \). In the third information structure, we divide the middle of the race into three pieces: \( B^3_i(k_i, k_{-i}) = \{k_i\} \times \{1, 2\} \) if \( k_{-i} \in \{1, 2\} \), \( B^3_i(k_i, k_{-i}) = \{k_i\} \times \{3, 4\} \) if \( k_{-i} \in \{3, 4\} \) and \( B^3_i(k_i, k_{-i}) = \{k_i\} \times \{5, 6\} \) if \( k_{-i} \in \{5, 6\} \). In the fourth and last information structure, each firm knows exactly the other firm’s development stage: \( B^4_i(k_i, k_{-i}) = \{k_i, k_{-i}\} \). In addition, at each information structure both firms know if the other is at the beginning of the race or at the end. That is, at state \( k = 0 \) we have \( B^4_i(k_i = 0, k_{-i} = 0) = \{0, 0\} \) and at \( k_{-i} = 7 \) we have \( B^4_i(k_i, k_{-i} = 7) = \{k_i, 7\} \).

Each firm’s belief about the other firm’s development stage is the uniform distribution over the observable event \( B_i \). For example, suppose \( k_1 = 1, k_2 = 4 \) and firms have the second information structure. Then, \( B^2_i(1, 4) = \{1\} \times \{4, 5, 6\} \), \( B^2_i(1, 4) = \{4\} \times \{1, 2, 3\} \), \( \lambda^2_i(\{4, 5, 6\})(k_2) = \frac{1}{3} \) for each \( k_2 \in \{4, 5, 6\} \) and \( \lambda^2_i(\{1, 2, 3\})(k_2) = \frac{1}{3} \) for each \( k_1 \in \{1, 2, 3\} \).

Given an investment level \( x_i \), the probability that firm \( i \) will advance from state \( k_i (k_i > 1 \text{ and } k_i \neq 7) \) to \( k_i + 1 \) is \( Q_i(k_i + 1|x_i, k_i) = q_1 x_i + (1 - x_i)q_2 \). As the state \( k = 0 \) only means that the race is at the beginning, the probability that firm \( i \) will move to the second development stage \( k = 2 \) is \( Q_i(2|x_i, k_i = 0) = q_1 x_i + (1 - x_i)q_2 \). If the firm is at the end of the race, it remains there forever \( Q_i(k_i = 7|x_i, k_i = 7) = 1 \).
Firms can only advance one stage at a time; therefore, the probability that the firm will remain in the same stage is $Q_i(k_i|x_i, k_i) = 1 - Q_i(k_i + 1|x_i, k_i)$ for $k_i \neq 7$ and $Q_i(1|x_i, k_i = 0) = 1 - Q_i(2|x_i, k_i = 0)$ if $k_i = 0$. We fix $q_1 = .3$ and $q_2 = \frac{4}{7}$, so there is a positive probability that the firm will advance to the next development stage regardless of its investment level.

Note that the Markov law of motion $Q$ and the cost structure are the same for both firms. Therefore, the only source of asymmetry between firms in our numerical exercise is their information regarding the other firm’s development stage.

2.3.2 Numerical Analysis

In order to calculate equilibrium in pure Markov investment strategies for the R&D race, we adapt the Pakes and McGuire [43] iterative procedure. Before we start the calculation, we set the value function $V^0$ to the discounted monopoly profits at $k_i = 7$ and $k_{-i} \neq 7$ and the discounted duopoly profits for $k = \{7, 7\}$. Also, we set the initial investment strategy $x^0$ to zero.

In the first step we calculate the optimal strategy $x^1$ at each observable event given that the other firm follows $x^0$ and the value function is $V^0$. Then, the value function $V^1$ is equal to the expected profits when both firms employ the strategy $x^1$ and the value function is $V^0$. We repeat this procedure until $\{V^k, V^{k-1}\}$ and $\{x^{k+1}, x^k\}$ are close pointwise.\(^3\)

2.3.3 Strategies and Value Function

We start by examining the strategic interaction between firms on each information structure for a specific set of parameters. Tables 1-4 present the equilibrium investment strategies and tables 5-8 present firms’ value functions for the case where

\(^3\)We set the stopping criteria to $\epsilon = 10^{-6}$
there are no patents, one table for each information structure. In this simulation, the parameters of the model are $\mu = 0.25$, $\pi^M = 1$ and $\delta = 2$. We pick this specific set of parameters to guarantee that firm strategy is not degenerate, i.e. firms make the maximum possible investment at each state, and the value of the firm changes as they go along the race. For example, if the cost of investing were too low both firms would invest the maximum possible and the value function would be roughly the same at each information structure. Therefore there would be no possibility of interpreting equilibrium strategies. In Appendix C we present the optimal investment strategies and value function for the case where a patent that lasts forever is awarded to the first firm to complete the project.

The first characteristic of the optimal strategies is that firms invest more as they become closer to the end of the race. As investing in R&D is costly, when firms become closer to the end of the race, it becomes more likely that they will complete the project and start receiving profits in the product’s market. Hence, in our setup, research in the early steps receives less investment than product finishing. Note that changing the cost structure does not change this characteristic of equilibrium investment strategies because investing at later stages costs more.

In the first and second information structures, investment is decreasing in the other firm’s stage. Given that the other firm is closer to the end of the race, firms lower investment, because it is less likely that they will arrive at the end first and receive the monopolistic profits. Firms take into consideration the fact that if the other firm finishes first they have to share the product’s market and get the duopolistic profits.

In the third information structure, investment is not decreasing with the other firm’s stage in the 6th stage. At the 6th stage the firm is already very close to the end,
so if the other firm is at the beginning \( k_{-i} \in \{1, 2\} \), it is better to cut investments. Still, when the other firm moves to \( \{3, 4\} \), the firms increase investments. On the other hand, if the other firm is in the last event \( \{5, 6\} \), it is already too close to the end. Then, firms lower investment because they know that they would probably have to share the product’s market profit. In the fourth information structure, equilibrium strategy has the same characteristic, but this starts in the 4th stage.

Note that since we present the optimal strategy for the case where there are no patents and the market may end up as a duopoly, firms invest even when the other firm has already completed the project. Firms know that once they finish the project themselves, they get the duopolistic profits in the product’s market. Still, investment decreases significantly.

Firm value at each observable event follows the same pattern as investment strategy. It increases when firms become closer to the end of the race, decreases significantly when the other is at the end of the race, and also decreases when the other firm is closer to the end of the race. The only exception to this pattern is in the fourth information structure when \( k_i = \{6\} \). When the other firm moves from \( k_{-i} = \{1\} \) to \( k_{-i} = \{2\} \), firm value increases. This increase in the firm’s value follows from the fact that they invest more, so it becomes more likely that they will finish the race earlier and get the monopolistic profits for a larger number of periods.

In the other case where a patent is awarded to the first firm to arrive at the end of the race, the equilibrium strategy changes. Investment is increasing in the other firm’s stage when the firm is in the lead of the race. Then, it decreases when the other firm catches up and decreases further as the other firm advances toward the end of the race. This effect on strategies is driven by our “winner takes all” setup. Even if both firms complete the project, only the firm that finished first can explore
the product’s market. Also, firms cancel investment once they learn that the other completed the project.

\[
\begin{array}{|c|c|c|c|}
\hline
(k_i, P_i(k_{-i})) & \{0\} & \{1,2,3,4,5,6\} & \{7\} \\
\hline
0 & 0.07 & - & - \\
1 & - & 0.07 & 0.03 \\
2 & - & 0.10 & 0.04 \\
3 & - & 0.16 & 0.05 \\
4 & - & 0.40 & 0.13 \\
5 & - & 0.53 & 0.17 \\
6 & - & 0.67 & 0.22 \\
7 & - & 0.00 & 0.00 \\
\hline
\end{array}
\]

Table 2.1: Optimal Investment Strategy for Information Structure 1 - No Patents

\[
\begin{array}{|c|c|c|c|}
\hline
(k_i, P_i(k_{-i})) & \{0\} & \{1,2,3\} & \{4,5,6\} & \{7\} \\
\hline
0 & 0.09 & - & - & - \\
1 & - & 0.09 & 0.04 & 0.03 \\
2 & - & 0.14 & 0.06 & 0.04 \\
3 & - & 0.21 & 0.11 & 0.05 \\
4 & - & 0.46 & 0.31 & 0.13 \\
5 & - & 0.56 & 0.46 & 0.17 \\
6 & - & 0.65 & 0.64 & 0.22 \\
7 & - & 0.00 & 0.00 & 0.00 \\
\hline
\end{array}
\]

Table 2.2: Optimal Investment Strategy for Information Structure 2 - No Patents

\[
\begin{array}{|c|c|c|c|c|}
\hline
(k_i, P_i(k_{-i})) & \{0\} & \{1,2\} & \{3,4\} & \{5,6\} & \{7\} \\
\hline
0 & 0.10 & - & - & - & - \\
1 & - & 0.10 & 0.05 & 0.03 & 0.03 \\
2 & - & 0.16 & 0.09 & 0.05 & 0.04 \\
3 & - & 0.22 & 0.16 & 0.08 & 0.05 \\
4 & - & 0.48 & 0.44 & 0.25 & 0.13 \\
5 & - & 0.56 & 0.54 & 0.39 & 0.17 \\
6 & - & 0.64 & 0.66 & 0.60 & 0.22 \\
7 & - & 0.00 & 0.00 & 0.00 & 0.00 \\
\hline
\end{array}
\]

Table 2.3: Optimal Investment Strategy for Information Structure 3 - No Patents
<table>
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<tr>
<th>$(k_i, P_i(k_{-i}))$</th>
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Table 2.4: Optimal Investment Strategy for Information Structure 4 - No Patents

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Table 2.5: Firm’s Value Function for Information Structure 1 - No Patents

<table>
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</table>

Table 2.6: Firm’s Value Function for Information Structure 2 - No Patents

Table 2.7: Firm’s Value Function for Information Structure 3 - No Patents


Table 2.8: Firm’s Value Function for Information Structure 4 - No Patents

2.3.4 Comparative Statics on Model Parameters

In this section we study the effect of some key parameters of the model in the summary statistics of the race. We study the effect of investment costs (\( \delta \)), intensity of market competition (\( \mu \)) and market size (\( \alpha \)). The market size is a constant that multiplies monopolist profit.

For the comparative statics on each parameter, we present two sets of graphs. The first is the duration of the race before the first firm finishes the race and the market becomes a monopoly and the expected time before the market becomes a duopoly. Also, we present the expected time for the market to become a monopoly for the case where there are patents.
There is also a second set of graphs with welfare statistics. We present consumer surplus, firm value at the beginning of the race, and total welfare for both the no patents and the patents case in the equilibrium calculated. In each graph we plot the summary statistics for each information structure.

Together, firm strategy and the law of motion on the stage of development, imply a stochastic process in the stage of development of both firms. Then, we can use this fact to calculate the expected time until one firm finishes the race and the expected time for both firms to finish the race (for the no patents case).

The duration of the race has a direct impact on consumer welfare. During the time where the product does not exist, consumers get zero surplus. Therefore, lowering the time to get the first invention increases consumer welfare. In addition, for the case where there are no patents, the expected time before the second firm finishes the race also affects consumer welfare. When the market becomes a duopoly, firms are not able to perfectly coordinate and share equally the monopolistic profits, hence consumer surplus increase ($\pi^D = \mu \pi^M < \frac{\pi^M}{2}$).

As an example, Table 9 presents these summary statistics for the R&D race presented in the last section.

<table>
<thead>
<tr>
<th></th>
<th>No Patents</th>
<th>Patents</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Info 1</td>
<td>Info 2</td>
</tr>
<tr>
<td>Duration to Monopoly</td>
<td>25.65</td>
<td>25.19</td>
</tr>
<tr>
<td>Duration to Duopoly</td>
<td>393.45</td>
<td>365.06</td>
</tr>
<tr>
<td>Consumer Surplus</td>
<td>7.63</td>
<td>7.74</td>
</tr>
<tr>
<td>Firm’s Value</td>
<td>2.17</td>
<td>2.84</td>
</tr>
</tbody>
</table>

Table 2.9: Summary Statistics for $\mu = 0.25$, $\pi^M = 1$ and $\delta = 2$.

In this specific R&D race, as firms know more about the other firm’s development stage, the time before the first invention decreases. With less information about the
other firm’s development stage, firms invest less. Investing in R&D is costly. Hence, if it turns out that the other firm has already completed the project and there are no patents, the other firm keeps investing, but a smaller amount. Then, the time before both firms finish the product is much higher than the time it takes for the first firm to complete the product.

Lowering the time before the first firm completes the project increases consumer surplus. Therefore, as firms have more information about the other firm’s stage, consumer surplus is higher. Also, firm value increases when firms have more information. More information is positive for firms because they are better able to assess the cost and benefits of investment. Hence, total welfare follows the same pattern.

Comparing consumer surplus for the no patents and the patents case, we get the standard argument in favor of patents. For this set of parameters, offering patent protection lowers the expected time to project completion. This gain in consumer welfare from lowering the expected duration of the race is higher than the benefits of having a second firm in the product’s market. Still, in the patents case, firm value decreases significantly. In the strong patent regime that we consider in this paper, all investment is thrown away if the other firm finishes the project first. The overall effect on welfare of a patent regime is negative, but it decreases as firms have more information. For this example, a strong patent is worse for society, especially if firms don’t have very good information about the other firm’s development stage.

2.3.5 The Effect of Market Size

We now discuss the effect of market size $\alpha$ on the race’s summary statistics for each information setup and patent regime. There are three sets of figures. Figure 1 presents the effect of market size on the duration of the race and Figure 2 the effect on consumer surplus, firm value and total welfare. Figure 3 presents the
difference between the race statistics in the no patents and the patents case. For this calculation, we fix the other parameters of the model to $\mu = 0.25$, $\pi^M = 1$ and $\delta = 2$.

The first thing to notice is that the duration of the race is decreasing with $\alpha$ for every possible setup. When we increase the size of the market, and therefore the product’s market profits, firms have a greater incentive to invest. In addition, when firms invest more in R&D, it becomes more likely that they will finish the product earlier.

The effect of $\alpha$ on consumer surplus, firm value and total welfare is also positive. With a larger market size, firms have incentive to invest more and complete the product earlier. Then, consumers are able to buy the product earlier resulting in a greater consumer surplus. Still, in the no a patent regime, the difference between consumer surplus increases with $\alpha$.

In Figure 3 we present the difference between the race’s statistics for the no patents and the patents case. For very small values of $\alpha \leq 0.2$ the no patents regime yields lower expected time to monopoly. When we increase $\alpha$, the patents regime yields lower race duration. The effect on consumer surplus is also not monotonic. For $\alpha < 2.1$ the patents regime yields higher consumer surplus, whereas above this value, the no patents regime is better. Also, the impact on consumer surplus of having a strong patent regime is greater the more informed firms are. Firm value increases with $\alpha$ and, in terms of total welfare, the no patents regime is better. The only exception is the complete information structure when $\alpha \in [1.4, 1.82]$. 
Figure 2.1: R&D race duration - Market Size
Figure 2.2: Consumer Surplus, Firm Value and Total Welfare - Market Size
Figure 2.3: Difference between race’s statistics by patent regime - Market Size
2.3.6 The Effect of Investment Cost

We now turn to the effect of investment cost, $\delta$, on the R&D race. In Figure 4, we present the effect of $\delta$ on the duration of the race, Figure 5 presents its effect on consumer welfare, firm value and total welfare, and Figure 6 presents the difference between the summary statistics on the no patents and the patents case for each information structure. We fix the other model’s parameters to $\mu = 0.25$, $\pi^M = 1$ and $\alpha = 1$.

In this race, the investment cost profile is the same for both firms and equal to $c = (\delta, \delta, \delta, 1, 1)$. Hence, the parameter $\delta$ is the cost of investing in R&D in the first half of the project. A larger $\delta$ implies that both firms have higher development costs in the first half of the race, as well as a higher cost asymmetry between the first and the second half of the race.

When we increase $\delta$, we also increase the duration of the race. When we increase the cost of investing in R&D, firms lower investment, resulting in higher expected time to complete the race. Also, when firms have more information about their opponent’s stage, the duration of the race is lower.

Since increasing $\delta$ results in higher expected duration of the race, it also lowers consumer surplus. Still, the impact of $\delta$ on firm value is small for all information structures. The change in total investment from a marginal increase in $\delta$ is very small when compared to the discounted monopolistic profits in the product’s market. Moreover, the impact of $\delta$ on total welfare is negative.

The patents regime lowers the expected duration of the race, with the exception of the first information structure and small values of $\delta < 1.75$. Therefore, consumer surplus is also higher with a patents regime. The impact of the patent regime on the race duration and consumer surplus is greater when firms have more information.
and tend to decrease with $\delta$.

Firm values, on the other hand, are greater in the no patents regime and the impact of patents is greater when firms have less information. The resulting impact of the patents regime on total welfare is negative and the difference is bigger when firms are less informed.

### 2.3.7 The Effect of Market Competition

In the last part of this section we examine the effect of the intensity in the product’s market competition $\mu$. In Figure 7 we present the effect of $\mu$ on the duration of the race and in Figure 8 the effect on consumer surplus, firm value and total welfare. We fix the other model’s parameters to $\pi^M = 1$, $\alpha = 1$ and $\delta = 2$.

The parameter $\mu$ captures the intensity of competition in the product’s market duopoly. A lower value of $\mu$ implies a tougher competition, where $\mu = .5$ implies a perfectly collusive setup where firms share the monopolistic profit equally. In our simple patent setup, the first firm to complete the project explores the market forever. Therefore it is only meaningful to study the impact of $\mu$ for the case where there are no patents.

Changing $\mu$ has two main effects on the R&D race. First it affects a firm’s incentive to keep investing once the other firm has already completed the project. Hence, investment increases with $\mu$. On the other hand, when the other firm increases investment it is not necessarily optimal to increase investment too. It depends on the benefit of getting the monopolistic profits for a longer period with the higher investment cost during the race.

When we combine these two effects of $\mu$ we get a non-monotonic impact on race duration. For lower values of $\mu$ the duration of the race to reaching monopoly is increasing, but as it approaches $.5$ the duration of the race decreases. Also, the
Figure 2.4: R&D race duration - Investment Costs
Figure 2.5: Consumer Surplus, Firm Value and Total Welfare - Investment Costs
Figure 2.6: Difference between race's statistics by patent regime - Investment Costs
difference in the race’s duration between information structures decreases with $\mu$.

The non-monotonic impact of $\mu$ on the race’s duration translates into a non-monotonic impact on consumer surplus. Still, firm value increases with $\mu$ as firms are able to get higher profits in the duopoly in the product’s market. The impact of $\mu$ on total welfare is positive. The eventual loss in consumer surplus is overcome by the increase in firm value. In addition, the difference in consumer surplus, firm value and total welfare decreases with $\mu$.

\section{Conclusion}

In this paper we present and show equilibrium existence of a multistage R&D race model where firms need to complete a finite number of intermediate steps to complete the product or project. Also, during the race, firms may not know the competitor’s development stage.

We simulate a version of the model and show the impact of market size, investment costs and intensity of market competition on the race. We show that the impact of these variables is not the same for each information structure. The impact is greater when firms are more informed and the product’s market is large and the impact is smaller when investment costs are high.

When we increase investment costs, race duration increases, while firm value, consumer surplus and total welfare all decrease. On the other hand, increasing the market size has the opposite effect. Lowering the intensity of competition in the product’s market increases firm value and total welfare. Still, its effect on consumer surplus depends on its specific value. Having a strong patent regime is better in terms of consumer surplus only when the market is small. Firm value and total welfare decreases with patents.
Figure 2.7: R&D race duration - Competition Intensity
Figure 2.8: Consumer Surplus, Firm Value and Total Welfare - Competition Intensity
In this paper we do not study the impact of cost asymmetries, information asymmetries, and other types of patent regimes (licensing etc.). These are interesting and important questions but are left to future work.
Chapter 3

A Note on Uniqueness of Bayesian Nash Equilibrium for Supermodular Incomplete Information Games

3.1 Introduction

The objective of this note is to provide sufficient conditions under which a supermodular incomplete information game has a unique Bayesian Nash equilibrium.

We call a static incomplete information game supermodular, whenever the utility function satisfies supermodularity conditions and the action space is a lattice. This class of games has been extensively studied in the literature, see for example Milgrom and Roberts [37], Vives [50], Van Zandt [51], etc. The main characteristic of this class of games is that there exists some type of complementarity between players’ actions. For example, in a Cournot competition game, if one firm increases the quantity produced the other firms’ best response is to do the same and increase its own production.

To obtain the uniqueness result we make additional restrictions on player’s beliefs, and on the utility function, besides supermodularity and increasing differences.
We suppose player’s beliefs have finite support, there are no irrelevant types; the utility function is concave and satisfies strict diagonal dominance. The no irrelevant types is a regularity assumption. It means that there is no type of other players that is considered impossible by all types of each player. Without this assumption, one step of the uniqueness proof is not well defined.

We also need two assumptions on the utility function. The first assumption, is that the utility function is concave. The second assumption is strict diagonal dominance. This assumption requires the first order derivative of the player’s utility function over their own action to be more affected by a change in their own action than the sum of the impact of all other players’ actions. Several authors use the strict diagonal dominance condition to show uniqueness in complete information games, for example Gabay and Moulin [18] and [37].

The method of proof for the uniqueness theorem extends Gabay and Moulin [18]’s proof for complete information games to supermodular incomplete information games. The first step in the proof, is to show that the expected utility preserves the same properties of the utility function. The second and main step of the proof, is to show that the best response function is a weak contraction.

Uniqueness proofs are useful, because in the class of games with unique equilibrium, comparative statics are not ambiguous. That is, in games with we cannot, ex-ante, point out which equilibrium will arise in the game once an exogenous variable changes.

There are few equilibrium uniqueness result for incomplete information games in the literature. The most recent work is from Mathevet [32]. The author shows when we can guarantee equilibrium uniqueness for global games. On the other hand, Mason and Valentinyi [31]’s objective is to provide sufficient conditions for existence
and uniqueness of monotone pure strategies.

This note is composed by two sections. In the first section, we present the model, the sufficient conditions for uniqueness of interim Bayesian Nash equilibrium and prove the uniqueness theorem. In the second and last section we provide two applications of the main theorem, one for an arms race game under incomplete information and the other for a Cournot competition game under incomplete information about costs and linear demand.

3.2 Framework and Main Result

There is a finite set of players $N = \{1, \cdots, n\}$ facing uncertainty about the state of the world $s \in S$. We assume $S$ is compact Polish, e.g. separable and completely metrizable. The interactive uncertainty about $S$ is described by a type space, which is a tuple $(S, N, (T_i, \lambda_i)_{i \in N})$. $\lambda_i : T_i \to \Delta(S \times T_{-i})$ is a mapping, and $\Delta(S \times T_{-i})$ is the set of probability distributions over $S \times T_{-i}$. We call a point $t_i \in T_i$ a type. The set $T_i$ contains all possible types of player $i$ and we endow it with its Borel $\sigma$-algebra. We call $\lambda_i$ the belief mapping. It associates a probability distribution over $S$ and the type of other players for each possible type of player $i$. Since $S$ is a compact Polish space, it follows from Mertens and Zamir [?] and Brandenburger and Dekel [8] the existence of an universal type space\footnote{A type space is universal whenever every type space can be embedded into it. See the references above for details.} from which $T = \times T_i$ is a subset. Under the assumptions in this paper player’s beliefs may not necessarily come from a common prior.

Given a probability distribution $F$ over a product space $X \times Y$, we define $\text{supp} F$ as the smallest closed set $A \subset X \times Y$ such that $F(A) = 1$. Furthermore, we define $\text{mrg}_X F$ and $\text{mrg}_Y F$ to be the marginal distribution of $F$ with respect to $X$ and $Y$.
respectively.

**Assumption 8** \( \lambda_i(t_i) \) has finite support for each \( t_i \in T_i \) and \( i \in N \).

**Assumption 9** The type space \( T = \times T_i \) satisfies the no irrelevant type property:

\[
\cup_{t_i \in T_i} \text{supp mrg}_{T-i} \lambda_i(t_i) = T_{-i}
\]

for each \( i \in N \).

Under Assumption 2, there is no type profile of other players that is considered impossible by all types of player \( i \) for each \( i \in N \).

Each player has a set of actions \( A_i \) available. To simplify notation, we assume \( A_i \) is a compact subset of \( \mathbb{R} \). However, all results in this paper remain valid for the case where \( A_i \) is a compact metric lattice\(^2\).

Let \( u_i : S \times T_i \times A_i \rightarrow \mathbb{R} \) be a bounded, \( S \times T \)-measurable mapping. We call \( u_i \) an utility function for player \( i \).

**Assumption 10** The utility function \( u_i \) is concave in \( a_i \), twice continuously differentiable with respect to \( A_i \) and has increasing differences with respect to \((a_i, a_{-i})\), e.g.

\[
\frac{\partial^2 u_i}{\partial a_i \partial a_j} \geq 0 \text{ for } i \neq j.
\]

In the general case where \( A_i \) is a compact metric lattice we also need \( u_i \) to be supermodular. Note that supermodularity is trivially satisfied whenever the function’s domain is unidimensional.

**Assumption 11** The utility function \( u_i \) satisfies strict diagonal dominance if

\[
\left| \frac{\partial^2 u_i}{\partial a_i^2} (t_i, s, a) \right| > \sum_{j \neq i} \left| \frac{\partial^2 u_i}{\partial a_i \partial a_j} (t_i, s, a) \right|
\]

for each \((t_i, s, a) \in T_i \times S \times A_i \).

\(^2\)A compact metric lattice is a lattice with a compact metrizable topology such that the lattice operations are continuous.
A pure strategy for player $i$ is a measurable mapping $\sigma_i : T_i \rightarrow A_i$. It associates an action to be played by each possible type of player $i$. We denote the space of strategies for player $i$ by $\Sigma_i$ and of all players by $\Sigma = \times \Sigma_i$. As usual, when we refer to all players except $i$, we add the subscript $-i$.

The interim expect utility is a mapping $h_i : T_i \times S \times \Sigma \rightarrow \mathbb{R}$ as follows:

$$h_i(t_i, \sigma) = \int_{S \times T_{-i}} \lambda_i(t_i)(s, t_{-i})u_i(t_i, s, \sigma(t_i, t_{-i}))dsdt_{-i}. \quad (3.1)$$

A supermodular incomplete information game $\Gamma$ is a tuple $(N, S, (T_i, \lambda_i, A_i, u_i)_{i \in N})$ satisfying Assumption 3. An interim Bayesian Nash equilibrium is a strategy profile $\sigma^* \in \Sigma$ such that $\sigma^*_i(t_i) \in \arg \max_{a_i \in A_i} h_i(t_i, a_i; \sigma^*_{-i})$ for each $t_i \in T_i$ and $i \in N$.

**Definition 11** Let $f : X \rightarrow X$ be a function over a metric space $X$. If $d_X(f(x), f(y)) < d_X(x, y)$ for each $x, y \in X$ with $x \neq y$, then we call $f$ a weak contraction.

In order to get uniqueness of Bayesian Nash equilibrium, it is enough to show that the best response function is a weak contraction. To get the intuition behind this result, consider the complete information setup where a strategy for player $i$ is a point $a_i \in A_i$. Let $br_i : A_{-i} \rightarrow A_i$ be the best response function for player $i$ and $br = (br_1, \cdots, br_n)$ the best response function of all players. Suppose for a complete information game, the best response is a weak contraction. In addition, suppose by way of contradiction that $a^1 \neq a^2$ are equilibrium for this game. Then, since $a^1 = br(a^1)$ and $a^2 = br(a^2)$, we have $\max_i |a^1_i - a^2_i| = \max_i |br(a_{-i})^1 - br(a_{-i})^2| < \max_i |a^1_i - a^2_i|$, which is a contradiction to the fact that both strategies are equilibrium for this game. The second main step of the proof is to show that the best response for the incomplete information game is a weak contraction.

**Theorem 11** There exists a unique interim Bayesian Nash equilibrium for a supermodular incomplete information game satisfying Assumptions 1-4.
Proof. Since the game satisfies Assumption 3, we can guarantee equilibrium existence using Van Zandt [51]. Then, it remains to show that under assumptions 1 to 4 this interim Bayesian Nash equilibrium is unique.

The proof is divided in two main steps. In the first we show that the expected utility is concave and satisfies strict diagonal dominance. In the second step, we show that the best response function is a weak contraction.

Let \( a_{-i}(\sigma_{-i}) = (a_{t_{-i}k_{-i}}, \ldots, a_{tK_{-i}}) \) be a matrix where \( a_{t_{-i}k_{-i}} = \sigma_{-i}(t_{-i}k_{-i}) \) for each \( t_{-i}k_{-i} \in supp \ mrg_{T_{-i}} \lambda_i(t_i) \). We can interpret the entry \( a_{t_{-i}k_{-i}} \) as the vector of actions all players except \( i \) would choose if their true type were \( t_{-i}k_{-i} \). It follows from assumption 1 that this matrix has a finite number of columns, because there are only a finite number of types in the support of \( \lambda_i(t_i) \).

We can rewrite the expected utility of type \( t_i \) as a mapping \( v_{t_i} : A_i \times A_{-i}^{[supp \ mrg_{T_{-i}} \lambda_i(t_i)]} \rightarrow \mathbb{R} \) as follows:

\[
v_{t_i}(a_i, a_{-i}(\sigma_{-i})) = \sum_{(s,t_{-i}k_{-i}) \in S \times \{supp \ mrg_{T_{-i}} \lambda_i(t_i)\}} \lambda(t_i)(s, t_{-i}k_{-i}) u_i(t_i, s, a_i, a_{-i}, s_{-i}k_{-i}).
\]

Since \( u_i \) is concave and \( \lambda(t_i)(s, t_{-i}k_{-i}) \geq 0 \) the function \( v_{t_i} \) is also concave in \( a_i \). Next we show that \( v_{t_i} \) satisfies strict diagonal dominance with respect to \( (a_i, a_{-i}(\sigma_{-i})) \).

Consider the following inequalities:

\[
\left| \frac{\partial^2 u_i}{\partial a_i^2}(t_i, s, a_i, a_{-i}, t_{-i}k_{-i}) \right| > \sum_{j \neq i} \left| \frac{\partial^2 u_i}{\partial a_i \partial a_j}(t_i, s, a_i, a_{-i}, t_{-i}k_{-i}) \right| \\
\lambda_i(t_i)(s, t_{-i}k_{-i}) \left| \frac{\partial^2 u_i}{\partial a_i^2}(t_i, s, a_i, a_{-i}, t_{-i}k_{-i}) \right| \geq \sum_{j \neq i} \lambda_i(t_i)(s, t_{-i}k_{-i}) \left| \frac{\partial^2 u_i}{\partial a_i \partial a_j}(t_i, s, a_i, a_{-i}, t_{-i}k_{-i}) \right| \\
\sum_{(s,t_{-i}k_{-i})} \lambda_i(t_i)(s, t_{-i}k_{-i}) \left| \frac{\partial^2 u_i}{\partial a_i^2}(t_i, s, a_i, a_{-i}, t_{-i}k_{-i}) \right| > \sum_{j \neq i} \sum_{(s,t_{-i}k_{-i})} \lambda_i(t_i)(s, t_{-i}k_{-i}) \left| \frac{\partial^2 u_i}{\partial a_i \partial a_j}(t_i, s, a_i, a_{-i}, t_{-i}k_{-i}) \right| \\
\left| \frac{\partial^2 v_{t_i}}{\partial a_i^2}(a_i, a_{-i}(\sigma_{-i})) \right| > \sum_{j \neq i} \sum_{t_{-i}k_{-i}} \left| \frac{\partial^2 v_{t_i}}{\partial a_i \partial a_j}(a_i, a_{-i}(\sigma_{-i})) \right| \quad (3.2)
\]
The first inequality is the definition of strict diagonal dominance for $u_i$. We obtain the second inequality by multiplying by $\lambda_i(t_i)(t_{-i})^k$, which is bigger than zero if $(s, t_{-i})$ is in the support of $\lambda_i(t_i)$. To get the third inequality, we sum over all types of other players in the support of $\lambda_i(t_i)$. The last inequality is the definition of strict diagonal dominance for $v_{t_i}$.

Now we turn to the second step. Suppose by way of contradiction that the set of interim Bayesian Nash equilibrium is not a singleton. For instance, suppose $\sigma^1$ and $\sigma^2$ are equilibrium for this game.

Pick a player $i \in N$. There are two cases. Either there exists $t_{-i} \in T_i$ such that $\sigma^1_{-i}(t_{-i}) \neq \sigma^2_{-i}(t_{-i})$ or $\sigma^1_i(t_i) \neq \sigma^2_i(t_i)$ for some $t_i \in T_i$.

Suppose the first case. Then, from assumption 2 there exists some type $t_i \in T_i$ such that $t_{-i} \in \text{supp } mrg_{T_{-i}}\lambda_i(t_i)$ and $\sigma^1_{-i}(t_{-i}) \neq \sigma^2_{-i}(t_{-i})$.

Define $br_i(\sigma_{-i}) = \{\sigma_i \in \Sigma_i | \sigma_i(t_i) = \arg \max_{a_i \in A_i} v_{t_i}(a_i, a_{-i}(\sigma_{-i})) \forall t_i \in T_i\}$ as the best response for player $i$ and $br(\sigma) = (br_1, \cdots, br_N)$ the best response of all players. The best response $br_i$ is a function because for each type $t_i$ there exists a unique action that maximizes interim expected utility. Recall that the expected utility is concave and strict diagonal dominance implies that $\frac{\partial^2 v_i}{\partial a_i^2}(\cdot) > 0$.

Let $a^*_1 = br_i(\sigma^1_{-i})(t_i)$, $a^*_2 = br_i(\sigma^2_{-i})(t_i)$. The case in which $a^*_2 = a^*_1$ is trivial. For instance $|br_i(\sigma^1_{-i})(t_i) - br_i(\sigma^2_{-i})(t_i)| = 0 < \max_{j \neq i, t_j} |\sigma^1_j(t_j) - \sigma^2_j(t_j)|$. Now suppose $|a^*_2 - a^*_1| > 0$.

Optimality of $a^*_1$ and $a^*_2$ implies that $\frac{\partial v_i}{\partial a_i}(a^*_1, a^1_{-i}) = \frac{\partial v_i}{\partial a_i}(a^*_2, a^2_{-i}) = 0$. Let $a^1 = (a^*_1, a_{-i}(\sigma^1_{-i}))$ and $a^2 = (a^*_2, a_{-i}(\sigma^2_{-i}))$. Let $\theta \in [0, 1]$, and define $\Psi : [0, 1] \to \mathbb{R}$ as

$$\Psi(\theta) = \frac{\partial v_{t_i}}{\partial a_i}[a^1 + \theta(a^2 - a^1)].$$

It follows from the assumption that $u_i$ is twice continuously differentiable that $\Psi$ is also continuously differentiable. In addition, $\Psi(0) = \Psi(1) = 0$. Hence using Rolle’s
Theorem\(^3\) there exists \(\theta^* \in (0, 1)\) such that

\[
\Psi'(\theta^*) = \sum_{t_i \neq i} \sum_{j \neq i} \frac{\partial^2 v_{i_t}}{\partial a_i \partial a_j t_k} [a^1 + \theta^*(a^2 - a^1)](a_{j,t}^2 - a_{j,t}^1) + \frac{\partial^2 v_{i_t}}{\partial a_i^2} [a^1 + \theta^*(a^2 - a^1)](a_{j,t}^2 - a_{j,t}^1) = 0;
\]

hence,

\[
\frac{\partial^2 v_{i_t}}{\partial a_i^2} [a^1 + \theta^*(a^2 - a^1)](a_{j,t}^2 - a_{j,t}^1) = - \sum_{t_i \neq i} \sum_{j \neq i} \frac{\partial^2 v_{i_t}}{\partial a_i \partial a_j t_k} [a^1 + \theta^*(a^2 - a^1)](a_{j,t}^2 - a_{j,t}^1).
\]

Combining equations (2) and (3) we get

\[
\sum_{t_i \neq i} \sum_{j \neq i} \left| \frac{\partial^2 v_{i_t}}{\partial a_i \partial a_j t_k} [a^1 + \theta^*(a^2 - a^1)] \right| |a_{j,t}^2 - a_{j,t}^1| < \left| \frac{\partial^2 v_{i_t}}{\partial a_i^2} [a^1 + \theta^*(a^2 - a^1)] \right| |a_{j,t}^2 - a_{j,t}^1| \leq (3.4)
\]

\[
\leq \sum_{t_i \neq i} \sum_{j \neq i} \left| \frac{\partial^2 v_{i_t}}{\partial a_i \partial a_j t_k} [a^1 + \theta^*(a^2 - a^1)] \right| |a_{j,t}^2 - a_{j,t}^1| \leq (3.5)
\]

since \(a_1^* \neq a_2^*\). Hence,

\[
|a_{j,t}^2 - a_{j,t}^1| < \max_{t_i \neq i} |\sigma_1^j(t_j) - \sigma_2^j(t_j)| = \max_{t_i \neq i} |\sigma_1^i(t_i) - \sigma_2^i(t_i)|.
\]

(3.6)

In the second case \(t_i\) is such that \(\sigma_1^i(t_{-i}) = \sigma_2^i(t_{-i})\) for each \(t_{-i} \in \text{supp}\, \text{mrg}_{t_{-i}}\lambda_i(t_i)\).

Then, it must be that \(br_i(\sigma_1^i)(t_i) = br_i(\sigma_2^i)(t_i)\) because \(br_i(\cdot)\) has an unique maximizer. Therefore

\[
|br_i(\sigma_1^i)(t_i) - br_i(\sigma_2^i)(t_i)| = 0 < \max_{t_i} |\sigma_1^i(t_i) - \sigma_2^i(t_i)|
\]

(3.7)

for each \(t_i \in T_i\) since \(\sigma^1 \neq \sigma^2\).

Define \(\|\sigma^1 - \sigma^2\|_\infty = \max_{t_i} |\sigma_1^i(t_i) - \sigma_2^i(t_i)|\). As the same argument holds for every type \(t_i\) and every player \(i\) we have:

\[
\|\sigma^1 - \sigma^2\|_\infty = \|br(\sigma^1) - br(\sigma^2)\|_\infty < \|\sigma^1 - \sigma^2\|_\infty
\]

a contradiction. \(\blacksquare\)

\(^3\)The Rolle's theorem states that if a function is continuous on a closed interval \([a, b]\) and differentiable on the open interval \((a, b)\), then there exists \(c \in (a, b)\) such that \(f'(c) = 0\).
Corollary 12 Suppose there exists an interim Bayesian Nash equilibrium for an incomplete information game satisfying assumptions 1,2,4 where the utility function is concave in $a_i$ and twice continuously differentiable with respect to $A$. Then, the equilibrium is unique.

3.3 Example: Arms Race under Incomplete Information

This example is based on Milgrom and Roberts [37]. Suppose there are two countries engaged in arms race. Each country has an initial arms stock level of $y_n \in [0, y_{\text{max}}]$, which is privately known. Their beliefs about $y_{-n}$ are summarized by $\lambda_n(y_n)$. This game is static and they choose an arms level $x_n \in [0, x_{\text{max}}]$. Strategy is a mapping $\sigma_n : [0, y_{\text{max}}] \rightarrow [0, x_{\text{max}}]$ and ex-post payoff given by:

$$ f_n(y_n, x_n, x_{-n}) = -C(x_n + y_n) + B(x_n + y_n - x_{-n} - y_{-n}). $$

The function $C(\cdot)$ is smooth strictly concave and $B(\cdot)$ is smooth concave. In addition, whenever

$$ |-C''(x_n + y_n) + B''(x_n + y_n - x_{-n} - y_{-n})| > |-B''(x_n + y_n - x_{-n} - y_{-n})| $$

this game satisfies strict diagonal dominance and if beliefs satisfy Assumption 2 the equilibrium is unique.

3.4 Example: Cournot Competition under Incomplete Information

There are two firms in a Cournot competition. Each firm chooses the quantity to be produced $q_i \in [0, M]$, where $M$ is a large number. Firms profit is given by
\(u_i = q_i(\theta_i - q_i - q_j),\) where the other firms do not observe its cost of production \(\theta_i.\)

Hence, firm’s belief about \(\theta_{-i}\) is summarized by a probability measure \(\lambda_i.\)

We cannot apply the uniqueness theorem for this game, because it does not have increasing differences (\(\frac{\partial^2 u_i}{\partial q_i \partial q_j} = -1\)). Still, we know an equilibrium exists for this game. In fact, firms profit is concave and it satisfies the strict diagonal dominance condition:

\[
| - 2 | = \left| \frac{\partial^2 u_i}{\partial q_i^2} \right| > \left| \frac{\partial^2 u_i}{\partial q_i \partial q_j} \right| = | - 1 |.
\]

(3.8)

Whenever \(\lambda_i\) satisfies Assumption 3, we can apply the corollary to show that there exists a unique interim Nash equilibrium for this game.
Chapter 4

Strategic Equivalence Between Arbitrary Finite Normal Form Games

4.1 Introduction

Starting at least with Rapoport and Guyer [46] there has been some articles in the economics literature dealing with the problem of classifying normal form games. The objective of this literature is to propose meaningful ways to distinguish the infinite number of games that exist. Games that share the same property are grouped into equivalence classes.

There are several approaches to the question of what is a meaningful way to classify games. We can cite as an example the best response equivalence proposed by Morris and Ui [49] and Mertens [33] and the semi-algebraic approach by Germano [19].

Morris and Ui [49] consider two finite normal form games to be equivalent if they have the same best response correspondence. On the other hand, in Mertens [33] two games are equivalent if there exists a correspondence between pure strategy sets such that the best response correspondences are the same. We should note, however, that Mertens [33] objective is to ask whether a solution concept make the same prediction for two games with the same payoff convex hull, i.e. it is immune to duplication of
pure strategies or addition of strategies that are the convex combination of others, the ordinality property\(^1\).

The semi-algebraic approach by Germano [19] takes a different perspective on the problem of classifying games. Given a semi-algebraic\(^2\) solution concept, two games are equivalent with respect to this solution concept if there is a continuum path on the space of games between them such that the equilibrium correspondence is continuum along the path. The discontinuities of the equilibrium correspondence divide the space of finite games with the same number of strategies into equivalence classes.

One weakness shared by the two approaches described above is that they cannot handle games with different number of pure strategy sets.

We propose an alternative approach to classify normal form games that depends on the solution concept used and is able to handle games with different number of strategies.

Before we explain the procedure to assign an equivalence relation to a solution concept we need a few definitions. Fix a solution concept and a finite normal form game. Each point in the set of equilibria induces a preference relation over the set of pure strategies of each player. A pure strategy \(s_i\) is preferred to \(s'_i\) for player \(i\) at the equilibrium strategy profile \(\sigma^* = (\sigma_1^*, ..., \sigma_n^*)\) if the expected payoff for \(s_i\) is higher than \(s'_i\).

The definition of equivalence between games depends on the existence of a mapping between pure strategy sets with certain properties. We allow the possibility of

\(^1\)Mertens [33] is a deeper look at the invariance axiom, one of which Kohlberg and Mertens [15] require the stable equilibrium to satisfy.

\(^2\)A solution concept is semi-algebraic if it can be written as a first-order formula, i.e. involving variables and constants, the universal and existential quantifiers, the logical connectives, the operations (+, −, ×, \(\setminus\)) and the relations (=, <, >). See Blume and Zame [6] for further details.
the image of a pure strategy to be the empty set. The empty set represents the fact that a pure strategy may not have a correspondent in the other game.

A mapping is proper with respect to a preference profile if whenever a pure strategy is not mapped to the empty set, then all pure strategies which are at least as good are also not mapped to the empty set. One implication of this requirement on the equivalence relation is the it rules out the case where games with different number of pure strategy equilibrium are equivalent.

Fix a solution concept and consider the set of induced preference profile for two normal form games. These two games are equivalent if there exists a proper mapping between pure strategy sets such that the image of the set of induced preference profiles is equal to the set of induced preference profile in the other game. In addition, we require this mapping to be proper with respect to set of induced preference profile of the first game and the inverse to be proper with respect to the set of preference profile of the second game.

It is useful to let the classification system depend on the solution concept, because we can make the same prediction to all games in the same equivalence class. In our case two games in the same equivalence class have the same set of induced preference profile in equilibrium.

In proposition 1 we show that this binary relation just defined is reflexive, symmetric and transitive, so it is indeed an equivalence relation.

Game 1:

<table>
<thead>
<tr>
<th></th>
<th>$s_1^1$</th>
<th>$s_2^1$</th>
<th>$s_3^1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1^1$</td>
<td>1,1</td>
<td>-1, -1</td>
<td>2,0</td>
</tr>
<tr>
<td>$s_2^1$</td>
<td>0,-1</td>
<td>0,0</td>
<td>1,-2</td>
</tr>
<tr>
<td>$s_3^1$</td>
<td>-1,0</td>
<td>-2, 0</td>
<td>0,-2</td>
</tr>
</tbody>
</table>
Game 2:

<table>
<thead>
<tr>
<th></th>
<th>$s_1^2$</th>
<th>$s_2^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1^1$</td>
<td>1, -1</td>
<td>-1, -1</td>
</tr>
<tr>
<td>$s_2^1$</td>
<td>0, -1</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

For example consider games 1 and 2. The second game is obtained by the elimination of the strictly dominated strategy $s_3$ for both players. Since $s_3$ is strictly dominated it is never the most preferred pure strategy in any induced preference profile by a Nash equilibrium. The function mapping strategies 1 and 2 of the first game to strategies 1 and 2 of the second game respectively and the third strategy of the first game to the empty set is proper. Therefore, these two games are equivalent with respect to the Nash equilibrium.

Now consider game 3 below, which is not obtained from game 2 by the addition of strictly dominated strategies for each player. Our procedure determines that it is also equivalent to games 1 and 2 according to the Nash equilibrium.

Game 3:

<table>
<thead>
<tr>
<th></th>
<th>$s_1^2$</th>
<th>$s_2^2$</th>
<th>$s_3^2$</th>
<th>$s_4^2$</th>
<th>$s_5^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1^1$</td>
<td>3, 3</td>
<td>2, -4</td>
<td>-1, -2.9</td>
<td>2, 3</td>
<td>4, -4</td>
</tr>
<tr>
<td>$s_2^1$</td>
<td>-2.8, -2</td>
<td>-1, 2</td>
<td>-1, -2</td>
<td>2.5, 0</td>
<td>0.2</td>
</tr>
<tr>
<td>$s_3^1$</td>
<td>-1.9, -2</td>
<td>-0.5, -2</td>
<td>-1, -2</td>
<td>1.8, 0</td>
<td>0.1</td>
</tr>
<tr>
<td>$s_4^1$</td>
<td>-1, -5</td>
<td>2, 2</td>
<td>2, -1</td>
<td>4.3</td>
<td>1.0</td>
</tr>
<tr>
<td>$s_5^1$</td>
<td>-1.3</td>
<td>0, 2.5</td>
<td>-1.2, 8</td>
<td>5.5</td>
<td>2, -1</td>
</tr>
<tr>
<td>$s_6^1$</td>
<td>2, 2</td>
<td>2, 4</td>
<td>1, -1</td>
<td>1, 1</td>
<td>3.5</td>
</tr>
<tr>
<td>$s_7^1$</td>
<td>1, 2</td>
<td>-2, 3</td>
<td>-5, -1</td>
<td>1, -1</td>
<td>1, -1</td>
</tr>
<tr>
<td>$s_8^1$</td>
<td>0, 2</td>
<td>6, 2</td>
<td>3.2</td>
<td>3.3</td>
<td>-10, 0</td>
</tr>
</tbody>
</table>

In theorem 1 we show that the equivalence relation associated to a solution concept has finite number of equivalence classes. If we don’t restrict the maximum

---

3This game has 2 pure strategy equilibria: $(s_5^1, s_2^2), (s_1^1, s_1^2)$ and one mixed strategy involving both pure strategies.
number of pure strategies in the game there is at most a countable number of equivalence classes. Still, when there exists a maximum number of pure strategies, say \( K \) and \( N \) players, our procedure is able to distinguish between \( 2^{\frac{3NK(N-1)}{2}} \) classes.

In Section 2 we provide the basic notation and the main results. Some applications are presented in Section 3. The last section concludes.

### 4.2 Equivalence Between Finite Normal Form Games

Let \( \Gamma = (N, S, u) \) be a finite normal form game with a finite number of players \( N = \{1, ..., n\} \), a finite strategy set \( S = \times S_i \) and a set of payoff functions \( u = (\bar{u}_i|S)_i \). The finite pure strategy set \( S \) is a subset of \( \mathcal{S} \), the space of all possible strategies\(^4\). The space of all possible payoff functions over \( \mathcal{S} \) is denoted by \( \mathcal{U} \). The payoff function \( u_i : \mathcal{S} \rightarrow \mathbb{R} \) assigns a real number to each strategy profile \( s \in \mathcal{S} \).

We denote by \( S_{-i} = \times_{j \neq i} S_j \) the set of pure strategy of all players except \( i \). The space of probability distributions over \( S \) is denoted by \( \Delta(S) = \times \Delta(S_i) \). The space of probability distributions over strategies of all players except \( i \) is \( \Delta(S_{-i}) = \times_{j \neq i} \Delta(S_j) \).

The set of players is always fixed. For a given set of players \( N \), the space of games is denoted by \( \mathcal{G} = (\mathcal{S}, \mathcal{U}) \).

**Definition 12** A function \( E : \mathcal{G} \rightarrow 2^{\Delta(S)} \) is called a solution concept.

A solution concept is a function that assigns a subset of the mixed strategy space for each game. Some examples include the Nash, perfect and proper equilibrium.

A mixed strategy induces a preference relation on the set of pure strategies. Given \( \sigma \in \Delta(S) \), the mixed strategy of other players \( \sigma_{-i} \) can be viewed as player’s

\(^4\) Another approach is to take as primitive an outcome function that assigns a real number for each point in the strategy set and preferences that satisfies the Savage’s Axioms. Still, we adopt the standard position of taking the utility function and the strategy space as primitives.
i belief. Therefore, the induced preference relation over $S$ for player $i$ is defined as $s_i R_i (s_{-i}) s'_i$ if and only if $\sum_{s_{-i}} \sigma_{-i}(s_{-i}) u(s_i, s_{-i}) \geq \sum_{s_{-i}} \sigma_{-i}(s_{-i}) u(s'_i, s_{-i})$. The preference profile induced by $\sigma$ is denoted by $R(\sigma) = (R(\sigma_1), ..., R(\sigma_n))$. The space of preference profile over $S$ is denoted by $\mathcal{R}(S)$.

**Definition 13** The preference structure is a function $T : \Delta(S) \times U \rightarrow \mathcal{R}(S)$ that assigns the induced preference profile over $S$ by $\sigma$ for a payoff function in $U$.

For the sake of exposition, if the function $T$ is applied to a set of mixed strategies given by a solution concept at a particular game we omit the payoff function in the argument. That is, instead of writing $T[E(\Gamma), u]$ we write $T[E(\Gamma)]$.

A correspondence $\gamma : S \rightrightarrows S' \cup \{\emptyset\}$ is proper at a preference profile $R \in \mathcal{R}(S)$ if for each $s \in S$ such that $\gamma(s) \neq \emptyset$, then $\gamma(s') \neq \emptyset$ for each $s'Rs$.

From a correspondence $\gamma : S \rightrightarrows S' \cup \{\emptyset\}$ we can define the induced mapping $\overline{\gamma} : \mathcal{R}(S) \rightarrow \mathcal{R}(S') \cup \{\emptyset\}$ as follows: If $sRs'$, then $R'$ is such that $s'' R' s'''$ for each $s'' \in \gamma(s)$ and $s''' \in \gamma(s')$, whenever $\gamma(s)$ and $\gamma(s')$ are not empty.

**Definition 14** Two games $\Gamma = (S, u)$ and $\Gamma' = (S', u')$ are equivalent with respect to a solution concept $E$ if there exists a correspondence $\gamma : S \rightrightarrows S' \cup \{\emptyset\}$, non-empty, proper at $T(E(\Gamma))$, the inverse $\gamma^{-1}$ is proper at $T(E(\Gamma'))$ and

$$\overline{\gamma}(T[E(\Gamma)]) = T[E(\Gamma'), u'|_{\gamma(S)}]$$

Two games $\Gamma$ and $\Gamma'$ are considered to be equivalent with respect to a solution concept if there is a way to associate pure strategies from one game to the other such that the set of induced preference relations is the same over $\gamma(S)$. Note that in the definition of equivalence we don’t restrict the number of pure strategy profiles in $S$ or $S'$. 
Since $\gamma$ is non-empty and proper we guarantee that at least the most preferred pure strategies for the set of induced preference relations is mapped. In addition, both requirements for $\gamma^{-1}$ guarantee that $\gamma$ is surjective over the set of most preferred pure strategies for the second game.

If we don’t require the mapping to be proper in the definition of equivalence we may consider two games with different number of pure strategy equilibria to be equivalent.

Consider the following example in which the inverse mapping between pure strategy sets is not proper. This example also shows why we require it to be proper besides $\gamma$.

Take two games $\Gamma$ and $\Gamma'$ with two symmetric players and 2 pure strategies in the first and 3 in the second. Since this game is symmetric we omit the player’s index in the pure strategy. The pure strategy sets are: $S_i = \{s_1, s_2\}$ and $S'_i = \{s'_1, s'_2, s'_3\}$.

Let $\Gamma$ have one Nash equilibrium: $(s_1, s_1)$ with $R(s_1, s_1) = s_1 > s_2$. In addition, let $\Gamma'$ have two pure strategy Nash equilibria: $(s'_1, s'_1)$ with induced preference profile $R(s'_1, s'_1) = s'_1 > s'_2 > s'_3$ and $(s'_2, s'_2)$ with $R(s'_2, s'_2) = s'_2 > s'_1 > s'_3$. The second game is a 2x2 coordination game with one additional strictly dominated strategy.

Pick $\gamma$ to map $\gamma(s_1) = s'_1$ and $\gamma(s_2) = s'_3$. The mapping $\gamma$ is proper with respect to the induced preference by the equilibrium $(s_1, s_1)$, because $\gamma(s_1) \neq \emptyset$. On the other hand, $\gamma^{-1}$ is not proper with respect to $R(s'_2, s'_2) = s'_2 > s'_1 > s'_3$. By assumption, $\gamma^{-1}(s'_3) \neq \emptyset$, but $s'_2 R(s'_2, s'_2)s'_3$ and $\gamma^{-1}(s'_2) = \emptyset$.

By applying the induced mapping $\overline{\gamma}$ we get $\overline{\gamma}(s_1 > s_2) = s'_1 > s'_3$. If we restrict $s'_1 > s'_2 > s'_3$ and $s'_2 > s'_1 > s'_3$ to $\gamma^{-1}(S') = \{s'_1, s'_3\}$ we get $s'_1 > s'_3$. According to our definition these two games would be considered to be equivalent with respect to
the Nash equilibrium solution concept\(^5\).

**Example 13 (Equivalence Between two Games with Different Number of Strategies)**

Consider the following games:

\[
\Gamma = \begin{array}{c|cc}
L & R \\
T & 1,1 & -1,0 \\
B & 0,-1 & -1,-1 \\
\end{array}
\]

\[
\Gamma' = \begin{array}{c|ccc}
L & R & C \\
T & 1,1 & -1,0 & 0,-2 \\
B & 0,-1 & -1,-1 & -2,-2 \\
M & -2,-2 & -2,0 & -3,-3 \\
\end{array}
\]

Games \(\Gamma\) and \(\Gamma'\) are equivalent with respect to the perfect equilibrium solution concept. The perfect equilibrium mapping \(PE(\cdot)\) when applied to the first game yield \(PE(\Gamma) = (T, L)\) with ranking \(T[PE(\Gamma)] = (T > B, L > R)\).

Now consider the second game. The set of perfect equilibrium \(PE(\Gamma') = (T, L)\) and \(T[PE(\Gamma')] = (T > B > M, L > R > C)\).

Let \(\gamma\) be such that \(\gamma(T) = T, \gamma(B) = B, \gamma(L) = L\) and \(\gamma(R) = R\). This function is proper with respect to the any induced preference profile in \(S\), because it is injective. The inverse is well defined and it is proper with respect to the induced preference profile in the second game. The inverse maps the first and the second most preferred points in the strategy set for both players.

Applying the preference mapping \(\gamma\) induced by \(\gamma\) on the preference profile for the first game we get \((T > B, L > R)\). The set of induced preference profiles for the second game restricted to the points in \(\gamma^{-1}\) is \((T > B, L > R)\).

---

\(^5\)What is behind this counterintuitive result if we let the correspondence to be ordered is the loss of symmetry in the equivalence relation. It is straightforward to check that \(\Gamma_1 F_E \Gamma_2\), but not \(\Gamma_2 F_E \Gamma_1\).
Therefore these two games are equivalent with respect to the perfect equilibrium solution concept.

**Lemma 14** The binary relation $F_E$ associated to the solution concept $E$ is an equivalence relation i.e. it is reflexive, symmetric and transitive.

**Proof.** Fix $\Gamma = (S, u) \in G$. The first step is to show that $F_E$ is reflexive ($\Gamma F_E \Gamma$).

Take the mapping between pure strategy sets to be $\gamma(s) = s \forall s \in S$. This mapping is proper, hence $\Gamma F_E \Gamma$.

Next we show that the equivalence relation is symmetric, i.e. for $\Gamma = (S, u), \Gamma' = (S', u') \in G$, $\Gamma F_E \Gamma'$ implies $\Gamma' F_E \Gamma$. If $\Gamma$ and $\Gamma'$ are equivalent with respect to $E$, there exists a proper mapping $\gamma$ such that $\tau(T[E(\Gamma)]) = T[E(\Gamma'), u'|_{\gamma(S)}]$.

The inverse mapping $\gamma^{-1}$ is proper from the definition of equivalence. Hence, when we apply $\tau^{-1}$ in both sides we get $T[E(\Gamma), u|_{\gamma^{-1}(S')}] = \tau^{-1}(T[E(\Gamma'), u'|_{\gamma(S)}])$. Let $\gamma^{-1} = \psi$. The mapping $\psi : S' \rightarrow S$ is proper and $\tau(T[E(\Gamma')]) = T[E(\Gamma), u|_{\psi(S')}]$ is the definition of $\Gamma' F_E \Gamma$.

The last step is to show transitivity. Pick $\Gamma = (S, u), \Gamma = (S', u'), \Gamma'' = (S'', u'') \in G$. Suppose $\Gamma F_E \Gamma'$ and $\Gamma'' F_E \Gamma'''$. There exists $\gamma : S \rightarrow S' \cup \{\emptyset\}$ and $\psi : S' \rightarrow S'' \cup \{\emptyset\}$ proper such that

$$\tau(T[E(\Gamma)]) = T[E(\Gamma'''), u''|_{\gamma(S)}] \tag{4.1}$$

$$\tau(T[E(\Gamma')]) = T[E(\Gamma'''), u''|_{\gamma(S)}]. \tag{4.2}$$

By applying the inverse of $\psi$ in the second equality we get

$$T[E(\Gamma'''), u''|_{\gamma^{-1}(S''')}] = \psi^{-1}(T[E(\Gamma'''), u''|_{\gamma(S''')}]). \tag{4.3}$$

The only difference between equality (3) and (1) is over which subset of $S''$ the set of preference profile over $S''$ is defined. In equality (1), $T[E(\Gamma''), u''|_{\gamma(S)}] \subseteq \mathcal{R}(\gamma(S))$ and in (3), $\psi^{-1}(T[E(\Gamma'''), u''|_{\gamma(S''')}] \subseteq \mathcal{R}(\gamma^{-1}(S''')).$
These two sets of preference profiles are equal over $\psi^{-1}(S''') \cap \gamma(S)$, because $\psi^{-1}$ and $\gamma$ are proper. Properness also ensure that $\psi^{-1}(S''') \cap \gamma(S) \neq \emptyset$.

Hence, over $\psi^{-1}(S''') \cap \gamma(S)$,

$$
\overline{\psi}(T[E(\Gamma), u|_{\psi^{-1}(S''') \cap \gamma(S)}]) = \overline{\psi^{-1}}(T[E(\Gamma'''), u'''|_{\psi^{-1}(S''') \cap \gamma(S)}])
$$

This is a set of preference profile over $S'''$. When we apply $\overline{\psi}$ on both sides we get

$$
\overline{\psi}\overline{\gamma}(T[E(\Gamma), u|_{\psi^{-1}(S''') \cap \gamma(S)}]) = T[E(\Gamma'''), u'''|_{\psi^{-1}(S''') \cap \gamma(S)}] \quad (4.4)
$$

Now let $\pi : S \to \psi^{-1}(S''') \cap Im(\gamma) \cup \{\emptyset\}$ be defined as $\pi(s) = \psi(\gamma(s))$. When we rewrite (4) using $\pi$ we get

$$
\pi(T[E(\Gamma)]) = T[E(\Gamma'''), u'''|_{\pi(S)}] \quad (4.5)
$$

This is the definition of $\Gamma F_\pi \Gamma'''$ as we wanted to show. 

For the main result in this section we use Borel equivalence relations. An equivalence relation $F$ on a topological space $X$ is Borel if all the equivalence classes are Borel subsets of $X^2$, endowed with the product topology. For any two Borel equivalence relations $F$ on $X$ and $F'$ on $X'$, we say that $F$ is Borel reducible to $F'$, in symbols $F \leq_B F'$, if there exists an injection $f : X \to X'$ such that for $x \in X$, $x' \in X'$, $xFx' \iff f(x)F'f(x')$. The complexity of an equivalence relation $F$ is given by the number of Borel sets in the quotient space $X/F$. If $F \leq_B F'$ we say that the classification problem presented by $F$ to be at least as complex to the one presented by $F'$.

The proof of the last theorem is divided in two main parts. The first shows that any equivalence relation associated to a solution concept is at most as complex to the equality equivalence relation on the space of subsets of preferences. The second step shows that, in turn, the equality equivalence relation has countable equivalence
classes. The last assertion of the theorem follows from a fundamental result in the literature of Borel equivalence relations by Silver[48].

**Theorem 15** The equivalence relation $F_E$ associated to a solution concept $E$ is such that $F_E \leq_B \mathbb{N}$. Moreover, any other equivalence relation on $\mathcal{G}$ is either Borel reducible to $\mathbb{N}$ or else $F_E \geq_B \mathbb{R}$.

**Proof.**

A preference profile $R \in \mathcal{R}(S)$ for $S \subseteq \overline{S}$ can be represented as a finite list of symbols called a ranking. Each entry in this list corresponds to how some agent $i$ ranks a pair of strategies $s_i, s'_i \in S_i$. There are only three possibilities, either $s_iRs'_i$, $s'_iRs_i$ or both. If $s_i$ is strictly preferred than $s'_i$ the entry in the list representing the ranking between $s_i$ and $s'_i$ has the symbol $>$. If $s'_i$ is strictly preferred we put $<$ and if the player is indifferent between these two strategies we put $=$. The ranking induced by $R$ is denoted by $\lambda(R)$. The space of rankings over $S$ is denoted by $\Lambda(S)$.

Since the strategy set $S \subset \overline{S}$ for every game $\Gamma = (S, u)$ we can embed any preference profile $R \in \mathcal{R}(S)$ on $\mathcal{R}(\overline{S})$ by letting the players be indifferent among all strategies in $\overline{S}\setminus S$. Recall that in the definition of equivalence we use the preference structure function $T$ applied to the set of solutions for a solution concept $E$. The result is a set of preference profiles in $\mathcal{R}(S)$ which can be embedded in $\mathcal{R}(\overline{S})$. We can work on $\mathcal{R}(\overline{S})$ without loss of generality.

Consider the space of finite rankings $\Lambda(\overline{S})$ endowed with the discrete topology. Denote by $F(\Lambda(\overline{S}))$ the space of closed subsets of $\Lambda(\overline{S})$. We endow $F(\Lambda(\overline{S}))$ with the $\sigma$-algebra generated by $\{F \in F(\Lambda(\overline{S})) : F \cap U \neq \emptyset\}$ where $U$ varies over open subsets of $\Lambda(\overline{S})$. Let $=_{F(\Lambda(\overline{S}))}$ be the identity equivalence relation on $F(\Lambda(\overline{S}))$.

In the first step of the proof we show that for every equivalence relation $F_E$ associated to a solution concept $E$, we have $F_E \leq_B =_{F(\Lambda(\overline{S}))}$.
The saturation of any point \( x \in F(\Lambda(S)) \) with respect to an equivalence relation \( H \) is given by \([x]_H = \{ y \in F(\{>,<,=\}|xHy\}\). A Borel selector for \( H \) is a function \( s : F(\Lambda(S)) \to F(\Lambda(S)) \) such that \( xHy \Rightarrow s(x) = s(y)Hx \).

An equivalence relation \( H \) admits a Borel selection if every equivalence class is closed and the saturation of any open set is a Borel set (Kechris [25, thm. 12.16]). Both conditions are trivially satisfied in our case for any equivalence relation associated to a solution concept.

Fix the equivalence relation associated to a solution concept \( E \). Take \( s(\cdot) \) to be a Borel selector for \( F_E \). The quotient space \( F(\Lambda(S))\setminus F_E \) is given by \( \{ x \in F(\Lambda(S))|s(x) = x\} \). It is clear that \( \{ x \in F(\Lambda(S))|s(x) = x\} \subseteq F(\Lambda(S))\), hence \( F_E \leq_B = F(\Lambda(S)) \).

The next step is to show that \( =_{F(\Lambda(S))} \leq_B \mathbb{N} \). The space of finite rankings is countable. For a fixed number \( k \) of points in the strategy set it has \( 3^k \) elements. The power set of \( \Lambda(S) \) is also finite for a fixed \( k \). Since \( \Lambda(S) \) is countable, there exists a homeomorphism between \( F(\Lambda(S)) \) and \( \mathbb{N} \). The last assertion of the theorem is an application of Silver[48] in Kechris[26].

The main theorem in this section shows that the equivalence relation associated to a solution concept has at most a countable number of equivalence classes. It also shows that any other way of classifying games is either Borel reducible to an equivalence relation associated to a solution concept or else it has a continuum number of equivalence classes.

**Corollary 16** In the space of games with \( N \) players and at most \( K \) pure strategies, there exists at most \( 2^{3NK(K-1)/2} \) equivalence classes.

**Proof.** Pick \( =_{F(\Lambda(S))} \), the identity equivalence relation on the space of closed subsets of rankings over \( S \), \( F(\Lambda(S)) \).
To calculate the upper bound on the number of equivalence classes we need to count how many subsets there are in $F(\Lambda(\mathcal{S}))$.

Each ranking $\lambda \in \Lambda(\mathcal{S})$ is a finite list. To represent a preference over $K$ strategies we need $\frac{K(K-1)}{2}$ entries. With $K$ strategies there are $K^2$ combinations with two entries, but we don’t need $K$, because the preference relation is reflexive $s_i R s_i$ (each $s_i$ is indifferent to itself). We further reduce this number by half because the preference relation is symmetric ($s_i R s'_i$ implies not ($s'_i R s_i$)).

Each entry in the ranking can have 3 symbols ($>,<,\ldots$) and there are $N$ players, therefore we have $3^{\frac{NK(K-1)}{2}}$ possible rankings in $\Delta(\mathcal{S})$. In addition there are $2^{\frac{3NK(K-1)}{2}}$ subsets in the power set of $F(\Lambda(\mathcal{S}))$.

The last proposition shows that there exists an upper bound on the number of equivalence classes associated to a solution concept. For example, in all games with 2 strategies and 2 players, the procedure proposed in this article can distinguish between $2^6$ games. If there are at most 3 strategies for each player the procedure can distinguish between $2^{18}$ games.

### 4.3 The Nash Equilibrium Equivalence Class for up to 3x3 Games

In this section we use the equivalence relation induced by the Nash equilibrium to establish which games with two players and 2 or 3 strategies each are equivalent.

We first show which are the possible sets of induced preference profile using the Nash equilibrium set. Then, we show what are the games with each set of preference profile in equilibrium.

In table 1 we present the possible induced preference profiles in equilibrium for each pure strategy combination in the support of the equilibrium. In the first row
and first column is the set of induced preference profiles if \((s^1_1, s^2_1) \in \text{supp } E(\Gamma)\) for a given 2x2 game. If there exists a pure strategy equilibrium \((s^1_1, s^2_1)\), then at this equilibrium either player 1 strictly prefers \(s^1_1\) to \(s^1_2\) \((s^1_1 \succ P s^1_2)\) or the player is indifferent between these two strategies \((s^1_1 \sim s^1_2)\) and vice-versa for player 2.

| \(s^1_2\) | \(R^*(s^1_1, s^2_1) = (s^1_1 Rs^1_2, s^1_2 Rs^2_2)\) | \(R^*(s^1_1, s^2_2) = (s^1_1 Rs^1_2, s^2_2 Rs^1_2)\) |
| \(s^2_1\) | \(R^*(s^2_1, s^2_2) = (s^2_1 Rs^2_1, s^1_2 Rs^2_1)\) | \(R^*(s^2_2, s^2_1) = (s^2_1 Rs^1_1, s^2_2 Rs^1_2)\) |

Table 4.1: Possible induced preference profile in equilibrium for 2x2 games.

Suppose \(T[NE(\Gamma)] = (s^1_1 Ps^1_2, s^2_1 Ps^2_2)\) for some 2x2 game \(\Gamma\). Since \(s^1_1\) is strictly preferred to \(s^1_2\) for both players, then \(u(s^1_1, s^2_1) < u(s^1_1, s^2_2)\). On the other hand if \(T[NE(\Gamma)] = (s^1_1 Is^1_2, s^2_1 Is^2_2)\), then \(u(s^1_2, s^2_1) = u(s^1_1, s^2_1)\) or the game has only one completely mixed strategy equilibrium. From the induced preference relation we can say something about the payoff relationship between the strategies in the support.

Permutations on the pure strategy set do not change the set of Nash equilibria. Therefore there exists a proper mapping (in fact a permutation) between games in which the set of induced preference relations is a singleton. We can characterize games with only one induced preference profile using a preference in \(R^*(s^1_1, s^2_1)\).

The Nash equilibrium restricts the possible sets with two induced preference profiles. For example there is no game such that \(T[NE(\Gamma)] = \{(s^1_1 Ps^1_2, s^2_1 Ps^2_2) \cup (s^1_2 Ps^2_1, s^2_2 Ps^2_2)\}\). For both preference profiles to be possible it must be that \(u(s^1_1, s^2_1) > u(s^2_2, s^2_1)\) and \(u(s^1_1, s^2_1) < u(s^2_1, s^2_1)\) which is impossible. The only way we can have both \((s^1_1, s^2_1)\) and \((s^2_1, s^2_1)\) in the support of \(NE(\Gamma)\) is when \(s^1_1 Is^2_1\).

Still, the Nash equilibrium places no restriction on the set of possible induced preference profile in equilibrium with \(s^1_1\) and \(s^2_1\) in the support. We can have for example a 2x2 game with two pure strategy Nash equilibrium. In this case we have \(T[NE(\Gamma)] = \{(s^1_1 Rs^1_2, s^1_1 Rs^2_2) \cup (s^2_2 Rs^1_1, s^2_2 Rs^2_1)\}\). Any combination between
preference profiles in $R^*(s_1^1, s_1^1)$ and $R^*(s_1^2, s_1^2)$ is possible.

With the observations in the previous paragraphs we can count the number of equivalence classes in the 2x2 games with respect to the Nash solution concept. With only one induced preference in equilibrium we have 4, because the induced preference for each player can be either strict or indifferent, hence 2 possibilities for each player. With two induced preference profiles there are 16 more equivalence classes $4 \times 4$, the possible rankings for $(s_2^1, s_2^2)$.

Example 17 The Nash equilibrium equivalence classes 2x2 games

One induced preference profile:

\[
\begin{array}{c|c|c}
 & s_1^1 & s_1^2 \\
\hline s_1^1 & 1,1 & 1,0 \\
\hline s_1^2 & 0,0 & 0,1
\end{array}
\]

\[\Gamma_1(s_1 P s_2) = \begin{cases} s_1^1 & s_2^1 \\ s_1^2 & s_2^2 \end{cases} \]

\[\Gamma_2(s_1^1 P s_2^1, s_1^2 P s_2^2) = \begin{cases} s_1^1 & s_2^1 \\ s_1^2 & s_2^2 \end{cases} \]

\[\Gamma_3(s_1^1 I s_2^1, s_1^2 P s_2^2) = \begin{cases} s_1^1 & s_2^1 \\ s_1^2 & s_2^2 \end{cases} \]

\[\Gamma_4(s_1^1 I s_2^1, s_1^2 I s_2^2) = \begin{cases} s_1^1 & s_2^1 \\ s_1^2 & s_2^2 \end{cases} \]

Two induced preference profiles:

\[\Gamma_5(s_1 P s_2, s_2 P s_1) = \begin{cases} s_1^1 & s_2^1 \\ s_1^2 & s_2^2 \end{cases} \]

Example 18 Possible induced preference profile in equilibrium for 3x2 games.

\[\begin{array}{c|c|c}
 & s_1^1 & s_1^2 \\
\hline s_1^1 & R^*(s_1^1, s_1^1) = (s_1^1 Rs_2^1 & s_1^2 Rs_3^1, s_1^3 Rs_2^3) & R^*(s_1^1, s_1^2) = (s_1^1 Rs_2^2, s_2^2 Rs_1^2) \\
\hline s_1^2 & R^*(s_1^2, s_1^1) = (s_1^2 Rs_1^1 & s_1^3 Rs_3^1, s_1^3 Rs_2^2) & R^*(s_1^2, s_1^2) = (s_1^2 Rs_1^2, s_2^2 Rs_3^2) \\
\hline s_1^3 & R^*(s_1^3, s_1^1) = (s_1^3 Rs_2^1 & s_1^3 Rs_3^1, s_1^3 Rs_2^3) & R^*(s_1^3, s_1^2) = (s_1^3 Rs_1^2, s_2^2 Rs_3^1)
\end{array}\]

In the second part of this section we study 3x2 games. Table 3 presents the set of possible induced preference profiles for each pure strategy combination in the support of the equilibrium correspondence.
A game in which the support of the equilibrium correspondence is composed entirely by at most two strategies is equivalent to a 2x2 game. Suppose $T[NE(\Gamma)] = s_1^1Rs_2^1Ps_3^1, s_1^2Rs_2^2$). Define a mapping $\gamma(s_3^1, s_2^2) = \emptyset$ and $\gamma(s) = s$ otherwise. This mapping is proper for the 3x2 game in consideration and $\pi(T[NE(\Gamma)]) = s_1^1Rs_2^1, s_1^2Rs_2^2$)

We know from the previous discussion on 2x2 games that there exists a game such that $T[NE(\Gamma)] = s_1^1Rs_2^1, s_1^2Rs_2^2$).

The equivalence class of 3x2 and 2x2 games is composed by the equivalence classes of 2x2 games plus some equivalence classes specific to 3x2 games, i.e those with 3 points in the support of the Nash equilibrium.

The set of induced preferences consistent with $s_3$ in the support of the Nash equilibrium besides $s_1$ and $s_2$ is $R^*(s_3^1, s_2^2) = \{s_3^1Is_1^1Rs_2^1, s_1^2Rs_2^2\} \cup (s_3^1Is_2^1Rs_3^1, s_2^2Rs_1^1)\}$.

We can infer from a induced preference relation in $T[NE(\Gamma)]$ a relation between payoffs. If two strategies are indifferent then the payoff is the same or else this is a game with a mixed strategy equilibrium with these strategies. Strict preferences translates to strict payoff inequalities.

In total for 3x2 games there are 20 equivalence classes (the 2x2 case) plus 16 (the number of equivalence classes with more than 1 induced preference relation for the 2x2 case) times the number of equivalence relation in $R^*(s_3^1, s_2^2)$ which is equal to 8. There are $20 + 16 \times 8 = 148$ equivalence classes for 2x2 and 3x2 games with respect to the Nash equilibrium.

The 3x3 case is similar to 3x2 except that now the second agent has one more strategy. The set of preference profiles compatible with $(s_3^1, s_3^2)$ in the equilibrium support, in the addition to the other strategies is $R^*(s_3^1, s_3^2) = (s_3^1Rs_1^1&s_3^1Rs_2^1, s_3^2Rs_2^2&s_3^2Rs_3^2)$.

The set $R^*(s_3^1, s_3^2)$ has 16 possible preference combinations, therefore for games with up to 3 strategies for each player there are $20 + 16 \times 16 = 276$ equivalence classes.
4.4 Conclusion

In this article we introduced a procedure to assign an equivalence relation on the space of finite normal form games for a given solution concept. The procedure consists in calculating the induced preference profile over the set of pure strategies for each equilibrium and comparing it to other games. To say that two games are equivalent we require the existence of a mapping between pure strategy sets called proper. A mapping between pure strategy sets and the empty set is proper with respect to a preference profile if whenever a strategy is not mapped to the empty set all strategies which are at least as good are also not mapped to the empty set. We then say that whenever there exists a proper mapping between the pure strategy sets such that the set of induced preference profile is the same, then these two games are equivalent. Unlike previous approaches, this procedure is able to handle the case where two games have different number of pure strategies.

We show in lemma 2.1 that this binary relation on the space of games is indeed an equivalence relation. It is reflexive, symmetric and transitive. The main result of this paper shows that the number of equivalence classes for the equivalence relation associated to a solution concept is countable or if there exists an upper bound on the number of pure strategies it is finite. Corollary 2.1 shows that there exists at most $2^{3NK(K-1)/2}$ equivalence classes for any solution concept. In particular, there exists 276 equivalence classes with respect to the Nash equilibrium for games with up to 3 strategies and 2 players.

We believe that the procedure introduced to assign an equivalence relation to a solution concept can be extended to extensive form games. This is left for future work.
Bibliography


Appendix

Appendix A - Existence of Equilibrium in Behavioral Strategies

In this section we show that if the action space is finite we can guarantee existence of equilibrium in behavioral strategies. That is, players may randomize their action choice.

For the remainder of this section, $A_i$ is a finite set and unless otherwise stated, the other variables are as in the framework section.

**Definition 15** A Markov behavioral strategy for player $i$ is a mapping $\beta_i : B_i \times T_i \times A_i \to [0, 1]$ such that $\beta_i(B_i, t_i; \cdot) \in \Delta(A_i)$ for each $(B_i, t_i) \in B_i \times T_i$.

We call $\beta_i$ Markov because it depends only on the actual observable event for each type not on a larger history of observable events. Let $D_i$ be the space of Markov behavioral strategies for player $i$ and $D = \prod D_i$ the space of Markov behavioral strategies of all players.

**Definition 16** The expected payoff for player $i$ is a function $h_i : T_i \times B_i \times D \times V_i \to [0, 1]$ as follows:

$$h_i(t_i, B_i; \beta, v_i) = \sum_{t_{-i}, s, a} \left[ u_i(t_i, s, a) + \delta \sum_{s'} v_i(t_i, B_i(s')) Q(s'|a, s) \right] \times (5.6)$$

$$\times \beta(B_i \times B_{-i}(s), (t_i, t_{-i}), a) \pi_i(t_i)(s, t_{-i})|B_i$$ (5.7)
The expected payoff is the sum of the present period expected payoff and the expected future payoff. Since player $i$ may not know the other players’ types and the state of the world, we weight each possibility by his beliefs $\pi_i(t_i)$ and the strategy $\beta$.

**Definition 17** A pair $(\beta^*, v^*) \in \mathcal{D} \times \mathcal{V}$ is an equilibrium in Markov behavioral strategies for a dynamic game of incomplete information and Markov transitions if:

1. $\beta_i^* \in \arg\max_{\beta_i \in \mathcal{D}_i} h_i(t_i, B_i; (\beta_i, \beta_{-i}^*); v_i^*)$

2. $v_i^*(t_i, B_i) = \max_{\beta_i \in \mathcal{D}_i} h_i(t_i, B_i; (\beta_i, \beta_{-i}^*); v_i^*)$.

for each $t_i \in T_i$, $B_i \in \mathcal{B}_i$ and $i \in N$.

The first condition is that $\beta_i^*$ is optimal for each $(t_i, B_i)$ given that the other players follow $\beta_{-i}^*$ and the value function is $v_i^*$. The second condition requires $v_i^*$ to be consistent with the expected payoff in equilibrium.

The next lemma is an important intermediate step for the proof of equilibrium existence.

**Lemma 19** Fix $t_i \in T_i$ and $B_i \in \mathcal{B}_i$. The expected payoff $h_i(t_i, B_i; \beta, v_i)$ is continuous in $(\beta, v_i)$.

**Proof.** Define

$$\bar{u}_i(t_i, s, a; v_i) = u_i(t_i, s, a) + \delta \sum_{s'} v_i(t_i, B_i(s')) Q(a)(s'|s).$$

(5.8)

We can rewrite the expected payoff as:

$$h_i(t_i, B_i; \beta, v_i) = \sum_{t_{-i}, s, a} \bar{u}_i(t_i, s, a; v_i) \beta(B_i \times B_{-i}(s), (t_i, t_{-i}), a) \pi_i(t_i)[(s, t_{-i})|B_i]$$

(5.9)

Pick a sequence $\{\beta_k, v_{i,k}\} \rightarrow \{\beta, v_i\}$ pointwise.
Since \( T_{-i} \times S \times A \) is finite we have only to show that \( \tilde{u}_i(t_i, s, a; v_{i,k})\beta_k(B_i \times B_{-i}(s), (t_i, t_{-i}), a) \rightarrow \tilde{u}_i(t_i, s, a; v_i)\beta(B_i \times B_{-i}(s), (t_i, t_{-i}), a) \) for each \((t_{-i}, s, a) \in T_{-i} \times S \times A\).

By definition of \( \tilde{u}_i(t_i, s, a; v_{i,k}) \) only the second term depends on \( v_{i,k} \). Hence, from pointwise convergence of \( v_{i,k} \), \( v_{i,k}(B_i(s'))Q(a)(s'|s) \rightarrow v_i(B_i(s'))Q(a)(s'|s) \) for each \( s' \in S \). Therefore \( \delta \sum_{s'} v_{i,k}(B_i(s'))Q(a)(s'|s) \rightarrow \delta \sum_{s'} v_i(B_i(s'))Q(a)(s'|s) \).

Next, \( \beta_k(B_i \times B_{-i}(s), t, a) \rightarrow \beta(B_i \times B_{-i}(s), t, a) \) by assumption. Moreover, the expected payoff is a finite sum of convergent sequences in the real line therefore, it also converges. □

**Theorem 20** Equilibrium exists in Markov behavioral strategies for a dynamic game of incomplete information and Markov transition.

**Proof.** Define the correspondence \( \Phi : D \times V \rightarrow D \times V \) as \( \Phi(\beta, v) = (\times \Phi_\beta(\beta, v), \times \Phi_v(\beta, v)) \), where:

\[
\Phi_\beta^i(\beta, v) = \left\{ \beta_i \in D_i | \beta_i \in \arg \max_{\beta_i \in D_i} h_i(t_i, B_i; (\beta_i, \beta_{-i}); v_i) \forall (t_i, B_i) \in T_i \times B_i \right\}
\]

\[
\Phi_v^i(\beta, v) = \left\{ v_i \in V_i | v_i(t_i, B_i) = \max_{\beta_i \in D_i} h_i(t_i, B_i; (\beta_i, \beta_{-i}); v_i) \forall (t_i, B_i) \in T_i \times B_i \right\}
\]

Both the space of behavioral strategies and value functions are compact and convex because \( D_i = [0, 1]^{T_i \times B_i \times A_i} \) and \( V_i = [0, 1]^{T_i \times B_i} \).

From Lemma 1.1, the expected payoff for player \( i \) is continuous in \((\beta, v_i)\), hence, we can apply Berge’s maximum theorem\(^6\) to establish that \( \Phi_\beta^i \) is upper-hemicontinuous with non-empty and compact valued and \( \Phi_v^i \) is continuous. Moreover, the same properties hold for \( \Phi_\beta \) and \( \Phi_v \), because the product of upper-hemicontinuous correspondences is upper-hemicontinuous (Aliprantis and Border [7][17.28]).

\(^6\)Aliprantis and Border[17.31]
The correspondence $\Phi^i_\beta$ is convex valued, because the expected payoff is linear in $\beta$. In addition, $\Phi^i_\beta(\beta, v)$ is single valued. We can apply Kakutani-Fan-Glicksberg’s fixed point theorem to show that the set of fixed points of $\Phi(\beta, v)$ is not empty.

Appendix B - The Dynamic Type Space

In this section we provide a brief introduction to Conditional Probability Systems and the construction of a universal type space. That is, the space containing all possible belief hierarchies. This section is based on Battigalli and Siniscalchi [5][sections 2 and 3].

Let $S$ be a Polish space, i.e. separable and completely metrizable. Denote by $\mathcal{A}$ its Borel sigma-algebra. The space $S$ represents the possible states of the world. For example, $S$ can be the product space between players’ private states $\Omega$ and publicly observed or common states $Z$: $S = \Omega \times Z$. In applied work $\Omega$ can be thought as the costs of each firm to provide a good or service and $Z$ the publicly observable aggregate demand.

A non-empty, finite, or countable family of open and closed sets $\mathcal{B} \subset \mathcal{A}$ is called a family of conditioning events. Players may be uncertain about the true state of the world $s \in S$ and each set $B \in \mathcal{B}$ corresponds to an observable event or relevant hypothesis over the state space $S$.

**Definition 18** Given a Polish space $S$, its sigma-algebra $\mathcal{A}$ and a family of conditioning events $\mathcal{B}$ a conditional probability system is a mapping

$$\mu : \mathcal{A} \times \mathcal{B} \rightarrow [0, 1]$$

satisfying the following axioms:

**Axiom 21** For all $B \in \mathcal{B}$, $\mu(B|B) = 1$. 


Axiom 22 For all $B \in \mathcal{B}$, $\mu(\cdot | B)$ is a probability distribution on $(S, \mathcal{A})$.

Axiom 23 For all $A \in \mathcal{A}$, $B, C \in \mathcal{B}$, if $A \subset B \subset C$ then $\mu(A | B) \mu(B | C) = \mu(A | C)$.

Conditional probability systems have a long tradition in economics starting with Myerson’s [38] work on multistage communication games. He considered the case where $S$ is finite, $\mathcal{A} = 2^S$, $\mathcal{B} = 2^S \setminus \{\emptyset\}$.

This framework is especially useful in dealing with zero probability events. The work of Mclennan on foundations for sequential equilibrium is a good example of this feature of CPS.

The space of conditional probability systems over $S$ associated with a family of conditioning events $\mathcal{B}$ is $\Delta^{\mathcal{B}}(S)$. The space of probability distributions over a set $Y$ is denoted by $\Delta(Y)$. The marginal of a probability distribution over a product space $p \in \Delta(X \times Y)$ in any of its components, say $X$, is denoted by $\text{mrg}_X p$.

In a strategic situation where there is an uncertainty about a parameter or the other players’ payoff function, not only the player’s beliefs matter. The player must take into consideration the beliefs of the other player’s about his beliefs and so on. To avoid dealing with this infinite hierarchy of beliefs Harsanyi [23] introduced the notion of type. The work of Mertens and Zamir [35] formalized this idea and showed the existence of a universal type space, a type space in which any other type space is a subset.

The extension of Mertens and Zamir’s work on dynamic environments is due to Battigali and Siniscalchi [5] on which this section is based.

---

7 See also Myerson [38] for decision theoretic foundations for CPS

8 See J. Halpern [21]. For a complete treatment of alternative probability spaces used in Economics.
Start with the basic uncertainty space $S$. Each set $X^n$, $\mathcal{B}^n$ is defined using the recursive formulation:

$$X^0 = S, \mathcal{B}^0 = \mathcal{B}$$

$$\forall n \geq 0,$$

$$X^{n+1} = C(X^n) \equiv X^n \times \Delta(X^n)$$

$$\mathcal{B}^{n+1} = C(\mathcal{B}^n) \equiv \{ C \subset X^{n+1} : \exists B \in \mathcal{B}^n, C = B \times \Delta_{\mathcal{B}^n}(X^n) \}.$$ 

The space of infinite hierarchies of CPS is defined by $H = \prod_{n=0}^{\infty} \Delta_{\mathcal{B}^n}(X^n)$. For the type space construction we restrict our attention to a special type of hierarchy. An infinite hierarchy of CPS $t = (\mu_1, \mu_2, \mu_3, \ldots) \in H$ is coherent if for all $B \in \mathcal{B}, n \geq 1$:

$$\text{mrg}_{X^{n-1}} \mu^{n+1}(\cdot | C^n(B)) = \mu^n(\cdot | C^{n-1}(B))$$  \hspace{1cm} (5.11)

A belief hierarchy is coherent if two probability distributions do not contradict each other at some level. The set of coherent belief hierarchy is denoted by $H_c$. Battigali and Siniscalchi[5][proposition 1] guarantees the existence of a "canonical" homeomorphism between the set of coherent hierarchies to the set of CPS over the state space and the set of coherent hierarchies: $f : H_c \to \Delta_{\mathcal{B}}(S \times H_c)$.

Let $E$ be a measurable event and $B \in \mathcal{B}$. A player with CPS $\mu$ is certain of $E$ given $B$, if $\mu(E|B) = 1$.

Requiring beliefs to satisfy the coherency condition does not rule out the case in which a belief hierarchy assigns positive probability to a set of incoherent hierarchies of other players. To rule out this possibility we further restrict our attention to the
set of belief hierarchies satisfying common certainty of coherency:

\[ H^1_c = H_c, k \geq 2 \]

\[ H^k_c = \{ t \in H^{k-1}_c : \forall B \in \mathcal{B}, f_B(S \times H^{k-1}_c) = 1 \} \]

\[ Y = \cap_{k \geq 1} H^k_c \]

The set \( Y \) is called the universal type space.

**Theorem 24** There exists a continuous homeomorphism \( \pi = (\pi_B)_{B \in \mathcal{B}} : Y \to \Delta^\mathcal{B}(S \times Y) \).

**Proof.** See Battigalli and Siniscalchi [5][proposition 2]. ■

The only theorem in this section guarantees the existence of a continuous homeomorphism between the universal type space \( Y \) and \( \Delta^\mathcal{B}(S \times Y) \).

**Appendix C - Optimal Investment Strategies and Firm Value when there are Patents**

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Table 5.2: Optimal Investment Strategy for Information Structure 1 - Patents
<table>
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Table 5.3: Optimal Investment Strategy for Information Structure 2 - Patents

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Table 5.4: Optimal Investment Strategy for Information Structure 3 - Patents

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Table 5.5: Optimal Investment Strategies for Information Structure 4 - Patents
Table 5.6: Firm’s Value Function for Information Structure 1 - Patents

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<th>$(k_i, P_{-i}(k_{-i}))$</th>
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Table 5.7: Firm’s Value Function for Information Structure 2 - Patents

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Table 5.8: Firm’s Value Function for Information Structure 3 - Patents

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Table 5.9: Firm’s Value Function for Information Structure 4 - Patents