Electoral Competition with Policy-Motivated Candidates*

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Abstract

In the multi-dimensional spatial model of elections with policy-motivated candidates, we prove that the candidates must adopt the same policy platform in equilibrium and that, if the candidates' gradients point in different directions, they must locate at some voter's ideal point. If the number of voters is odd, such equilibria must exhibit a type of symmetry condition: it must be possible to pair some voters so that their gradients point in exactly opposite directions. When the number of dimensions is more than two, this condition is knife-edge. If the number of voters is even, such equilibria never exist.

1 Introduction

In the tradition of spatial modeling in positive political theory, majority-rule elections are often conceptualized as competition between two candidates who stake out positions in a space of policies, followed by the votes of voters who observe those policy platforms. If the candidates are purely office-motivated, if the policy space is one-dimensional, and if the preferences of voters are single-peaked, then the median voter theorem applies: there is a unique equilibrium, and in it both candidates adopt the median ideal point of the voting population as their platforms (cf. Downs (1957), Black (1958)). The property of the median ideal point behind this result is that it is defeated by no other platforms in majority voting. In higher dimensional spaces, such an undominated policy position is referred to as a “core point.” In fact, regardless of dimensionality, office-motivated candidates must locate at core points in equilibrium: if one candidate were to locate at a beatable position, the other would move to exploit that opportunity. We refer to this phenomenon as “core equivalence.” An implication is that, in the absence of a core point, there will be no (pure strategy) equilibrium of the game between the candidates.

As is well-known, the existence of a core point when the number of voters is odd entails a symmetry condition on voter preferences that is extremely restrictive in two or more dimensions: Plott (1967) shows that a core point must be the ideal point of some voter, and the gradients of the other voters’ utility functions must be paired so that, for every voter with a gradient pointing in one direction, there is exactly one voter whose gradient points in the opposite direction (see also McKelvey and Schofield (1987)). This necessary condition suggests that, for “most” specifications of voter preferences, core points — and therefore electoral equilibria — will fail to exist. When existence does obtain, it will be vulnerable to even slight variations in preferences (cf. Rubinstein (1979), Schofield (1983), Cox (1984), Le Breton (1987)). When the number of voters is even, the results are not so negative: a core point need not be the ideal point of a voter, the symmetry condition is no longer necessary, and the existence of core points may be robust to variations in preferences.

Clearly, aside from the assumption of an odd number of voters in the negative results on equilibrium existence, the assumption that candidates are office-motivated plays an important role: since candidates care only about winning the election, if one’s position can be beaten by any other

position, the opposing candidate has an unambiguous incentive to move to the majority-preferred position. If candidates care about policy, as voters do, then, because the majority-preferred position may have undesirable policy implications, many such moves will not be in the interest of the opponent. The candidates' incentives to move to majority-preferred positions will be mitigated, creating the possibility of equilibrium existence even in the absence of a core point, thereby avoiding the strong negative conclusions drawn under the assumption of office-motivation. Indeed, the situation is well-understood in one dimension: as long as the ideal points of the candidates are on opposite sides of the median voter, then the median remains the unique electoral equilibrium. But, of course, equilibrium existence is not an issue in the traditional one-dimensional model.

In higher dimensions, little is known about the existence of equilibria under policy motivation or their relationship to the core. A related question is whether the candidates can adopt distinct positions in equilibrium, or whether they must adopt identical positions, a phenomenon called "policy coincidence" or "policy convergence." Calvert (1985) provides part of the answer: assuming the existence of a core point and Euclidean voter preferences (i.e., circular indifference curves), the unique electoral equilibrium is for both candidates to locate at the core point. But these assumptions are quite restrictive, together implying that the majority preference relation is transitive and equal to the preference relation of the "core voter." Calvert conjectures (p.79) that, if the assumption of Euclidean preferences is weakened, then other types of equilibria, in which candidates do not locate at the core point, will be created. Furthermore, because his result is predicated on the existence of a core point, it does not inform us of the generic case, where there is no undominated position. In particular, the result leaves open the question of whether electoral equilibria exist more generally than core points and, if so, it leaves the task of finding conditions characterizing equilibrium points in the absence of a core.

We take up these questions in a general model, assuming that the utility functions of voters and candidates are differentiable and strictly quasi-concave. We show that, in any equilibrium with neither candidate at his/her own ideal point, the candidates must take identical policy positions. Since equilibria where one candidate is at his/her ideal point are somewhat exceptional, the phenomenon of policy coincidence is quite general. We next consider, following Calvert (1985), equilibria in which the candidates' gradients point in different directions, i.e., the candidates have distinct policy

preferences near the equilibrium point. We give an example in which three voters have non-Euclidean preferences, there is a unique core point, and there is an equilibrium in which the candidates do not locate at the core point. Thus, the core equivalence result of Calvert does not generalize to arbitrary differentiable, strictly quasi-concave voter utility functions. In that example, however, the candidates locate at one voter's ideal point and a type of symmetry on the voters' gradients holds: for every voter whose gradient lies between the candidates' gradients, there must be exactly one voter whose gradient points in exactly the opposite direction. We prove that those properties are in fact necessary conditions for equilibrium when the number of voters is odd. Though potentially restrictive, the symmetry condition does not imply the existence of a core point nor, generally, the kind of fragility of equilibria seen in models with office-motivated candidates. Indeed, we give a three-voter, two-dimensional example in which the core is empty; there exists an electoral equilibrium with policy-motivated candidates; and the equilibrium is robust to small changes in the preferences of voters and candidates. Thus, the negative conclusions drawn for office-motivated candidates do not carry over with full force.

In three or more dimensions, the existence of equilibria is considerably more precarious. Given any equilibrium with an odd number of voters, we show that, for every voter whose gradient does not lie on the plane spanned by the candidates' gradients, there must be exactly one voter whose gradient points in the opposite direction. In other words, if we remove the voters whose gradients lie on that plane, the equilibrium platform must be a core point of the modified majority voting game. Because the plane is a lower-dimensional subspace, we would not expect it to contain the gradients of all voters. Typically, therefore, we must have some pairs of voters with diametrically opposed gradients, a result suggesting that electoral equilibria will be rare and that, when existence does obtain, it will be vulnerable to even slight variations of voter or candidate preferences. Thus, with three or more dimensions and a finite, odd number of voters, the equilibria conjectured by Calvert (1985) can exist only in knife-edge situations.

When the number of voters is even, the optimistic case in models of office-motivated candidates, we show that equilibria in which the candidates' gradients point in different directions *never* exist, even in the one-dimensional case. Though the generality of this result may be unexpected, it is due to a feature of policy-motivation that is quite intuitive: a policy-motivated candidate may have an incentive to move from one platform to another, even if the

new platform does not beat that of his/her opponent, because that platform produces a better outcome in the event that a tie is broken in the candidate's favor. We give an example showing that, if the candidates' gradients are unrestricted, equilibria are indeed possible under some conditions (e.g., if it is an equilibrium for the candidates to locate at their ideal points). We end by giving conditions on the preferences of voters and candidates under which the restrictions we impose on candidates' gradients are non-binding: in many environments, the only possible equilibria are those in which the candidates' gradients point in different directions. Thus, with a few interesting qualifications, the results on equilibrium existence in the multi-dimensional spatial model with policy-motivated candidates are negative. And in contrast to models with office-motivated candidates, this conclusion is essentially independent of the number of voters.

The issue of policy-motivated candidates has been addressed in related literatures. In a model of probabilistic voting with policy-motivated candidates, Wittman (1977, 1983) has shown that the policy coincidence result known for office-motivated candidates (and which we have proved for deterministic voters) breaks down. Calvert (1985) shows that the extent of divergence of the candidates' platforms varies continuously with the amount of policy-motivation added to the objective functions of office-motivated candidates. In the literature on "citizen candidates" (cf. Osborne and Slivinski (1996), Besley and Coate (1997, 1998), Duggan (2000), Banks and Duggan (2000)), candidates are assumed, as are all voters, to possess policy preferences. But these models differ from the spatial model of elections in that candidates cannot commit to policies prior to an election; rather, office holders choose policies optimally given their preferences and, in some models, given the effects of policy choices on future electoral prospects. In contrast, our paper contributes to the understanding of the effects of policy motivation by maintaining the other basic assumptions, deterministic voting and commitment among them, of the spatial model.

The remainder of the paper is organized as follows. In Section 2, we present the model of elections with policy-motivated candidates. In Section 3, we state our results on necessary conditions for existence of equilibria of two types: equilibria in which neither candidate locates at his/her ideal point, and equilibria in which the candidates' gradients point in different directions. In Section 4, we give conditions under which there are no other equilibria. In Section 5, we continue our discussion, presenting several examples to illustrate our results. All proofs are contained in an appendix.

2 The Model

We consider two candidates, A and B , competing for the votes of an electorate, N , containing a number n of voters. We use the notation C for an arbitrary candidate and i, j, k , etc., for an arbitrary voter. Let X be a convex subset of d -dimensional Euclidean space, $\mathbb{R}^d$. The candidates simultaneously choose policy platforms from X , with candidate C 's platform denoted x_C . We use the notation x, y, z , etc., for arbitrary policies. Each voter i has a preference relation on X represented by a strictly quasi-concave, differentiable utility function $u_i: X \rightarrow \mathbb{R}$. Thus, if voter i has a utility-maximizing platform, it is unique. We call such a policy i 's *ideal point* and denote it $\tilde{x}_i$. For simplicity, we assume that $\nabla u_i(x) = 0$ if and only if x is voter i 's ideal point. We assume that no two voters have the same ideal point: $\nabla u_i(x) = \nabla u_j(x) = 0$ for no x, i , and $j \neq i$. We say voter i 's preferences are *Euclidean* if i has an ideal point $\tilde{x}_i$ and, for some strictly decreasing function $f: \mathbb{R}_+ \rightarrow \mathbb{R}$, $u_i(x) = f(\|x - \tilde{x}_i\|)$, i.e., voter i has circular indifference curves.

We use the notation R for weak majority preference, P for strict preference, and I for indifference: xRy if and only if $u_i(x) \geq u_i(y)$ for at least half of the voters; xPy if and only if $u_i(x) > u_i(y)$ for more than half of the voters (i.e., not yRx); and xIy if and only if xRy and yRx . In the appendix, we state a lemma on the “star-shapedness” of majority preferences: if xRy , then any point between x and y will be weakly majority-preferred to y , strictly so if the number of voters is odd. We define the *core* as the set of platforms x weakly majority-preferred to all other platforms: for all $y \in X$, xRy . If the number of voters is odd, then a standard result under our assumptions is that the core, when non-empty, consists of a single point, say x^* , and that, for all $y \neq x^*$, x^*Py . Moreover, x^* is the ideal point of some voter, say i^* . In case all voters have Euclidean preferences, it is known that the majority preference relation coincides with the preferences of the “core voter” i^* , i.e., xRy if and only if $u_{i^*}(x) \geq u_{i^*}(y)$ (cf. Davis, DeGroot, and Hinich (1972)). Thus, in that case, the majority weak preference relation is complete and transitive, with circular indifference curves. None of these conclusions holds generally when n is even.

We assume each candidate C has a preference relation on X represented by a strictly quasi-concave, differentiable utility function $u_C: X \rightarrow \mathbb{R}$. If candidate C has an ideal point, we denote it by $\tilde{x}_C$. Again for simplicity, we assume $\nabla u_C(x) = 0$ if and only if x is candidate C 's ideal point. We assume that the candidates are policy-motivated, which means that a candidate may

face a tradeoff between desirable and successful policy platforms. Formally, we assume that each candidate C 's policy preferences, given the majority preference relation, induce a strict preference relation $\succ_C$ on platform pairs satisfying the following three minimal conditions.

- (C1) If zPy , if yPx or $u_A(z) > u_A(x)$, and if $u_A(z) > u_A(y)$, then $(z, y) \succ_A (x, y)$ (and likewise for B).
- (C2) If xRy and $u_A(y) > u_A(x)$, then $(y, y) \succ_A (x, y)$ (and likewise for B).
- (C3) If xIy , if $\{x_m\}$ is a sequence converging to x , if x_mPy for all m , and if $u_A(x) > u_A(y)$, then $(x_m, y) \succ_A (x, y)$ for large enough m (and likewise for B).

We also impose a condition that applies when n is even.

- (C4) If n is even, if xIy , if zRy , and if $u_A(z) > u_A(x)$, then $(z, y) \succ_A (x, y)$ (and likewise for B).

The intuition for these conditions is straightforward. Suppose that candidate A adopts platform x and candidate B adopts platform y , and consider condition (C1). Suppose candidate A can move to a platform z that will win for sure, and suppose that z is better than the platform used by candidate B . If z is also better than A 's initial platform x , or if the candidate was initially losing, then moving to z should be profitable, as condition (C1) stipulates. For condition (C2), if xRy , then we can think of x as the outcome of the election with some positive probability. If A prefers B 's platform y to x , and if A moves from x to y , then y is the outcome with probability one. This never makes A worse off and makes the candidate better off whenever x would have been the outcome before. Condition (C2) states that this is an improvement for A . Now consider condition (C3). If xIy , then we can think of y as being the outcome with some positive probability. If A prefers x to y and can beat y outright with platforms arbitrarily close to x , then, as stated in condition (C3), sufficiently small moves should be improvements for A . Condition (C4) also applies in case of a majority tie, but only when the number of voters is even. If xIy and A prefers a platform z that also weakly beats y , moving to z will produce a better outcome whenever the tie is broken in A 's favor. As long as the probability A wins in case of a tie is not influenced by this move, the move should be an improvement for A .

The preceding conditions characterize situations in which a candidate has an incentive to change his/her position. At times, we will want to use three additional conditions that are sufficient for the candidates to *not* have profitable deviations.

- (C5) If yPz or $u_A(z) \leq u_A(y)$, then not $(z, y) \succ_A (y, y)$ (and likewise for B).
- (C6) If candidate A has ideal point $\tilde{x}_A$ and if $\tilde{x}_APy$, then not $(z, y) \succ_A (\tilde{x}_A, y)$ (and likewise for B).
- (C7) If yPx , and if yPz or $u_A(z) \leq u_A(y)$, then not $(z, y) \succ_A (x, y)$ (and likewise for B).

Suppose that both candidates adopt platform y , which is then the outcome with probability one. If candidate A adopts a platform that loses to y , and therefore does not affect the policy outcome, or if A moves to some platform no more desirable than y , then that move should not be profitable, as stipulated in condition (C5). For condition (C6), if $\tilde{x}_APy$, so that candidate A wins by locating at his/her ideal point, then it is clear that no move can be profitable for A , as in condition (C6). Finally, if candidate A 's platform x loses to y , and if A moves to a platform z that also loses to y (and so does not change the policy outcome) or that is less desirable than y , then that move should not be profitable, as stated in condition (C7).

Conditions (C1)-(C7) hold, for example, in the following environment: voters vote sincerely (eliminating weakly dominated strategies), flipping coins when indifferent; the candidate with the majority of votes wins and implements his/her policy platform, with ties broken by a coin flip; and candidates evaluate lotteries over policy outcomes according to expected utility. In fact, we could allow voter i to randomize between the candidates with any positive probabilities when $u_i(x_A) = u_i(x_B)$. Conditions (C1)-(C3) hold — and so, therefore, does the analysis for n odd — even if these probabilities vary arbitrarily with the particular platforms over which i is indifferent. The conditions are general enough that we could even allow indifferent voters to abstain from voting with any probability (possibly one), as long as the winner in case of a tie is determined randomly with each candidate receiving positive probability.

We say (x_A, x_B) is an *equilibrium* if neither candidate C can deviate to a different platform to produce a preferred pair: there does not exist $x'_A \in X$ such that $(x'_A, x_B) \succ_A (x_A, x_B)$ (and likewise for B). We say that (x_A, x_B) is a

competitive equilibrium if x_A and x_B are interior to X and neither candidate's chosen platform is at his/her ideal point: $\nabla u_A(x_A) \neq 0$ and $\nabla u_B(x_B) \neq 0$. We say (x_A, x_B) is a *polarized equilibrium* if the platforms are interior and the candidates' gradients do not point in the same direction: there do not exist $\alpha, \beta \geq 0$, at least one non-zero, such that $\alpha \nabla u_A(x_A) = \beta \nabla u_B(x_B)$. Clearly, every polarized equilibrium is competitive.

3 Results

We first establish that, in every competitive equilibrium, the candidates must choose the same platform. Thus, despite the possibility that the candidates might have starkly different policy preferences, the incentives of electoral competition lead to a unique policy choice for the voters.

Theorem 1 *Assume (C1)-(C4). If (x_A, x_B) is a competitive equilibrium, then $x_A = x_B$.*

The theorem is stated only for competitive equilibria out of necessity, for, under conditions (C6) and (C7), it is a simple matter to construct non-competitive equilibria that violate policy coincidence. In Figure 1, for example, we assume one dimension and one voter with Euclidean preferences. Here, candidate A 's ideal point is to the left of candidate B 's, which is to the left of the voter's ideal point. If candidate B 's platform is his/her ideal point, $\tilde{x}_B$, and if candidate A locates anywhere to the left of B , then neither candidate can deviate profitably: since $\tilde{x}_B P x_A$, condition (C6) implies that B cannot gain by moving; and the only platforms to which A can move and influence the outcome of the election are worse than $\tilde{x}_B$, so (C7) applies. An example with n even can be constructed simply by placing a second voter's ideal point to the right of voter 1's. Proposition 1, in the next section, gives a condition that rules out the possibility of such equilibria when n is odd. In the one-dimensional case, the condition is simply that the candidates' ideal points lie on opposite sides of the median ideal point.

We now add the assumption that the gradients of the candidates do not point in the same direction. We establish that, when the number of voters is odd, the candidates must locate at some voter's ideal point, say $\hat{x}$. Moreover, a limited version of Plott's (1967) symmetry condition must hold: it must be possible to pair voters whose gradients are between the candidates' gradients with voters whose gradients point in exactly opposite directions. For vectors

Figure 1: A non-competitive equilibrium without policy coincidence

$p, q \in \mathbb{R}^d$, we use the notation $\text{cone}^\circ\{p, q\} = \{\alpha p + \beta q \mid \alpha, \beta > 0\}$ to denote the open cone generated by p and q .

Theorem 2 *Assume n is odd, and assume (C1)-(C3). If (x_A, x_B) is a polarized equilibrium, then $x_A = x_B = \hat{x}$, where $\nabla u_k(\hat{x}) = 0$ for some voter k . For $p \in \mathbb{R}^d$,*

$$|\{i \in N \mid \exists \alpha > 0 : \nabla u_i(\hat{x}) = \alpha p\}| = |\{i \in N \mid \exists \alpha < 0 : \nabla u_i(\hat{x}) = \alpha p\}|,$$

if either $\nabla u_A(\hat{x})$ and $\nabla u_B(\hat{x})$ are linearly dependent or $\nabla u_A(\hat{x})$ and $\nabla u_B(\hat{x})$ are linearly independent and $p \in \text{cone}^\circ\{\nabla u_A(\hat{x}), \nabla u_B(\hat{x})\}$.

Figure 2 depicts depicts a situation in which the necessary conditions given in the theorem are satisfied. Here, the candidates locate at voter 5's ideal point. The gradients of voters 1 and 3 point in opposite directions. The gradients of voters 2 and 4 are not matched in this way, but, because neither gradient (or its opposite) lies in the open cone generated by the candidates' gradients, the symmetry condition of the theorem is preserved.

By the first part of the theorem, the candidates must locate at some ideal point, say $\hat{x}$, in a polarized equilibrium. The proof of the second part of the theorem is largely concerned with the case in which the candidates' gradients are linearly independent. We show that the set of platforms weakly majority-preferred to $\hat{x}$, the region described by hash marks in Figure 3, lies below the hyperplanes defined by the gradients of the candidates. This implies a kind of "kink" in the boundary of that set, one that is not possible when the core is non-empty and the preferences of the voters are Euclidean. Under those conditions, the majority preference relation would coincide with the preference relation of the core voter, so the majority indifference curves would simply be circles and obviously could not have kinks. Thus, in Calvert's (1985) model, the only platform weakly preferred to $\hat{x}$ is $\hat{x}$ itself, i.e., the candidates must locate at the core point, and then symmetry of the voters' gradients follows from Plott's (1967) theorem. In the proof of Theorem 2,

Figure 2: The symmetry condition of Theorem 2

we show, without assuming Euclidean preferences or the existence of a core point, that the boundary of the set of platforms weakly majority-preferred to $\hat{x}$ is “kinked enough” only if the symmetry condition of the theorem holds.

That Theorem 2 does not hold generally for non-polarized equilibria, even those in which the candidates adopt the same platform, can be seen by modifying the example of Figure 1. Giving the candidates the same platform, say $\hat{x}$, anywhere between candidate B ’s ideal point and the voter’s, and assuming condition (C5), this is a competitive, non-polarized equilibrium: for each candidate, the only platforms weakly majority-preferred to $\hat{x}$ are less desirable than $\hat{x}$. Clearly, the candidates are not located at the ideal point of any voter, and the symmetry condition of the theorem is violated. Proposition 2, in the next section, gives a condition under which no such non-polarized equilibria will exist. In the one-dimensional case, the condition is simply that the candidates’ ideal points lie on opposite sides of the median.

The apparent implications of Theorem 2 are limited by the assumption that the number of voters is odd and by the domain of the symmetry condition: if the candidates’ gradients are linearly independent, it applies only

Figure 3: A kink in the boundary of the majority-preferred-to set

to voters with gradients in the open cone generated by the candidates' gradients. When the number of voters is odd and the dimension of the policy space is more than two, however, the result has implications much stronger than may at first seem. The next theorem says that, given an equilibrium $(\hat{x}, \hat{x})$, for every voter whose gradient does not lie on the plane spanned by the candidates' gradients, there must be a voter whose gradient points in exactly the opposite direction. Alternatively, if we delete the voters whose gradients lie on the plane spanned by the candidates' gradients, leaving the $\hat{x}$ voter, then the platform $\hat{x}$ must be a core point of the resulting majority preference relation.

Theorem 3 *Assume n is odd, and assume (C1)-(C3). If (x_A, x_B) is a polarized equilibrium, then $x_A = x_B = \hat{x}$, where $\nabla u_k(\hat{x}) = 0$ for some voter k . Moreover, for every $p \in \mathbb{R}^d$ such that $p \notin \text{span}\{\nabla u_A(\hat{x}), \nabla u_B(\hat{x})\}$,*

$$|\{i \in N \mid \exists \alpha > 0 : \nabla u_i(\hat{x}) = \alpha p\}| = |\{i \in N \mid \exists \alpha < 0 : \nabla u_i(\hat{x}) = \alpha p\}|.$$

To see that the theorem does not hold generally for non-polarized equilibria, consider Figure 4. Here we assume three voters and Euclidean preferences over a multi-dimensional policy space, with the ideal points of the voters arranged in an isosceles triangle, voter 1's ideal point at the apex. We put candidate B 's ideal point above that, and we put candidate A 's ideal point above that. Then, under condition (C5), it is a competitive equilibrium for both candidates to adopt the same platform anywhere between voter 1's and candidate B 's ideal points. Thus, the candidates locate at no voter's ideal point. Moreover, the span of the candidate's gradients is the line through their ideal points, and neither voter 2's nor voter 3's gradients can be opposed in the required way, violating the symmetry condition of the theorem.

The following corollary of Theorem 3 gives a general condition on the gradients of voters under which polarized equilibria fail to exist when n is odd. The condition holds quite widely when the dimension of the policy space is at least three. It suggests that, for "most" specifications of differentiable, strictly quasi-concave voter utility functions, we would not expect polarized equilibria to exist — and that, if existence did obtain, it would be sensitive to even slight variations of voter or candidate preferences.

Corollary 1 *Assume n is odd, and assume (C1)-(C3). Assume that, for every voter i possessing an ideal point, the dimension of $\text{span}\{\nabla u_j(\tilde{x}_i) \mid j \in N\}$ is at least three. And assume that, for all voters j and k , $\nabla u_j(\tilde{x}_i)$ and $\nabla u_k(\tilde{x}_i)$ are linearly independent. Then there does not exist a polarized equilibrium.*

Figure 4: A competitive non-polarized equilibrium violating the conditions of Theorem 3

The proof the corollary is simple. Theorem 3 tells us that, given a polarized equilibrium (x_A, x_B) , the candidates must locate at the ideal point of some voter, say i . Since $\text{span}\{\nabla u_A(\tilde{x}_i), \nabla u_B(\tilde{x}_i)\}$ is a two-dimensional space and the dimension of $\text{span}\{\nabla u_j(\tilde{x}_i) \mid j \in N\}$ is at least three, there is some voter j such that $\nabla u_j(\tilde{x}_i) \notin \text{span}\{\nabla u_A(\tilde{x}_i), \nabla u_B(\tilde{x}_i)\}$. But, under the assumptions of the corollary, there is no voter whose gradient points in the direction opposite that of voter j 's, a contradiction.

The implications of Theorem 2 when the number of voters is even are more striking, as that result allows us to prove the following.

Theorem 4 *Assume n is even, and assume (C1)-(C4). There does not exist a polarized equilibrium.*

In the proof of the theorem, we first verify that, as in Theorem 2, the candidates would have to locate at the ideal point, say $\hat{x}$, of some voter, say i . Deleting that voter from N , we are left with an electorate, N' , with an odd number of voters. Furthermore, there is no voter in N' with ideal point $\hat{x}$, violating a necessary condition in Theorem 2 for equilibrium in the reduced model. Thus, one of the candidates can move to a better platform, say x' , preferred by a majority of voters in N' to $\hat{x}$. Adding i back to the electorate,

x' still *weakly* beats $\hat{x}$. Under condition (C4), this still gives the candidate a profitable deviation, and we conclude that polarized equilibria cannot exist when n is even.

4 Other Equilibria

Theorem 1 established policy coincidence, but only for the class of competitive equilibria, and Theorems 2 through 4 concern only polarized equilibria. In this section, we present two results. The first gives a sufficient condition under which there are no equilibria other than competitive, and the second gives conditions under which there are no equilibria other than polarized. Proposition 1 uses the condition that, given either candidate's ideal point, there exists a majority-preferred platform that the other candidate also prefers. This extends the condition, frequently assumed in one-dimensional models, that the candidates' ideal points are on opposite sides of the median (or medians, when n is even). We discuss the plausibility of a stronger condition at the end of the section.

Proposition 1 *Assume (C1)-(C4). Assume that, if candidate A has an ideal point $\tilde{x}_A$, then there exists a platform $x \in X$ such that $xP\tilde{x}_A$ and $u_B(x) > u_B(\tilde{x}_A)$; likewise, if candidate B has an ideal point $\tilde{x}_B$, then there exists a platform $y \in X$ such that yPx_B and $u_A(y) > u_A(\tilde{x}_B)$. If (x_A, x_B) is an interior equilibrium, then either it is competitive or n is even and $x_A = \tilde{x}_A$ and $x_B = \tilde{x}_B$.*

This proposition leaves open the possibility of a non-competitive equilibrium in the n even case, as long as the candidates locate at their ideal points. This possibility is depicted in Figure 5, where the ideal points of the two voters are between those of the candidates. It is easy to see that the condition of Proposition 1 is satisfied: the ideal point of voter 1, $\tilde{x}_1$, is preferred to $\tilde{x}_A$ by both voters and by candidate B; similarly, $\tilde{x}_2$ is preferred to $\tilde{x}_B$ by the voters and by candidate A. Note that there exist open intervals Y and Z around $\tilde{x}_A$ and $\tilde{x}_B$, respectively, such that every platform in Y is majority-indifferent to every platform in Z . Thus, because there are no small moves for either candidate to platforms that will beat his/her opponent, our arguments (in the appendix) that one candidate will have a profitable deviation do not go through. Indeed, there is no compelling reason why one of the candidates

Figure 5: A non-competitive equilibrium with n even, as in Proposition 1

must have a profitable deviation in this situation — that will depend on the exact specification of the candidates' strategic preferences in the model.

The next proposition gives a condition, strengthening that of Proposition 1, under which all interior equilibria are polarized. Once again, the condition extends the familiar one from one-dimensional models that the candidates' ideal points are on opposite sides of the median. We will say that an interior platform x “fails the polarization condition” if $\alpha \nabla u_A(x) = \beta \nabla u_B(x)$ for some $\alpha, \beta \geq 0$, at least one non-zero.

Proposition 2 *Assume (C1)-(C4). Assume that, for each $x \in X$ failing the polarization condition, there exists a platform $y \in X$ such that yPx and, for some candidate C , $u_C(y) > u_C(x)$. If (x_A, x_B) is an interior competitive equilibrium, then it is polarized.*

The proof is trivial, owing to condition (C1), Theorem 1, and the strength of the condition stated in the proposition. To see that this condition is indeed stronger than that of Proposition 1, set $x = \tilde{x}_A$; then the condition of Proposition 2 yields C and y such that $u_C(y) > u_C(x)$; and then, of course, we must have $C = B$, fulfilling the condition of Proposition 1.

The condition of Proposition 2 is not completely transparent, and so it is of interest to understand when it (and, therefore, the condition of Proposition 1) might hold. Suppose that $d \geq 2$, that n is odd, and that voter preferences are Euclidean. Let $Y \subseteq X$ denote the *yolk*, the smallest closed ball intersecting all median hyperplanes (cf. McKelvey (1986)). Thus, if the hyperplane

$$H_{x,y} = \{z \in \mathbb{R}^d \mid 2z \cdot (x - y) = (x + y) \cdot (x - y)\}$$

bisecting two platforms, x and y , does not intersect Y , majority indifference between x and y cannot hold. Whether xPy or yPx depends on whether Y is on the x -side or y -side of H_{xy} . Suppose further that there exists $t \in \mathbb{R}^d$ such that, for all $y \in Y$,

$$t \cdot \tilde{x}_A < t \cdot y < t \cdot \tilde{x}_B,$$

Figure 6: The yolk “separating” the candidates’ ideal points

where $\tilde{x}_C$ is the ideal point of candidate C . For simplicity, we normalize t so that $\|t\| = 1$. Note that, since Y is compact, the minimum value of $t \cdot y$ over Y , denoted $\min t \cdot Y$, exists and $t \cdot \tilde{x}_A < \min t \cdot Y$. Likewise, $\max t \cdot Y < t \cdot \tilde{x}_B$. And note the implication that $t \cdot (\tilde{x}_B - \tilde{x}_A) > 0$. Obviously, this situation, depicted in Figure 6, is more plausible when the yolk is small, i.e., when the core is “close” to being non-empty. When the core is non-empty, it is equal to the yolk and the above condition holds as long as the candidates’ ideal points are not colinear with the core point.

When such a t exists, the assumption of Proposition 2 holds. To see this, note that the set of platforms that violate the polarization condition must lie on the line $\text{span}\{\tilde{x}_A - \tilde{x}_B\} + \tilde{x}_A$ spanned by the candidates’ ideal points, but not strictly between them. Letting x be such a platform, that means

$$x = \alpha \tilde{x}_A + (1 - \alpha) \tilde{x}_B = \tilde{x}_B + \alpha(\tilde{x}_A - \tilde{x}_B)$$

for $\alpha \geq 1$ or $\alpha \leq -1$. If $\alpha \geq 1$, then

$$\begin{aligned} t \cdot x &= t \cdot \tilde{x}_B + t \cdot (\tilde{x}_A - \tilde{x}_B) + (1 - \alpha)t \cdot (\tilde{x}_B - \tilde{x}_A) \\ &= t \cdot \tilde{x}_A + (1 - \alpha)t \cdot (\tilde{x}_B - \tilde{x}_A) \\ &\leq t \cdot \tilde{x}_A. \end{aligned}$$

Similarly, $t \cdot x \geq t \cdot \tilde{x}_B$ if $\alpha \leq -1$. (See Figure 6 for $\alpha \geq 1$.) Suppose, without loss of generality, that $\alpha \geq 1$. Define $x_\epsilon = x + \epsilon t$ for $\epsilon > 0$, and pick ϵ smaller than the lesser of

$$(\min t \cdot Y) - t \cdot \tilde{x}_A \quad \text{and} \quad t \cdot \tilde{x}_B - (\max t \cdot Y),$$

as in Figure 6. Then we have

$$\begin{aligned} t \cdot x_\epsilon &= t \cdot x + \epsilon \\ &\leq t \cdot \tilde{x}_A + \epsilon \\ &< \min t \cdot Y, \end{aligned}$$

which implies that the bisecting hyperplane H_{x, x_ϵ} does not intersect the yolk. And since the yolk is on the x_ϵ -side of the hyperplane, we have $x_\epsilon P x$. Finally, note that

$$\begin{aligned} t \cdot (\tilde{x}_B - x) &= \alpha t \cdot (\tilde{x}_B - \tilde{x}_A) \\ &> 0, \end{aligned}$$

which implies that $u_B(x_\epsilon) > u_B(x)$ for small enough ϵ , as required.

5 Discussion

We have shown that policy coincidence and a type of Plott symmetry are necessary conditions for equilibrium. In Figure 7, we verify that they are not sufficient for equilibrium. In this example, the voters have Euclidean preferences and the candidates are located at voter 3's ideal point. With candidate gradients as depicted, the conditions of Theorem 2 are satisfied, but either candidate can move to a more desirable platform preferred by voters 1 and 2, a profitable deviation by condition (C1). In this example, voter 2's ideal point is the core, a likely candidate for equilibrium. In fact, if the number of voters is odd, if condition (C5) holds, and if there is a core

Figure 7: Disequilibrium satisfying symmetry

point, it is easy to see that that it is an equilibrium for both candidates to locate at the core.

Calvert (1985) showed that, when voters' preferences are Euclidean and the core is non-empty, there can be no other polarized equilibria. To see how this core equivalence result can break down with non-Euclidean voter preferences, consider the example in Figure 8. We give voters 1 and 3 Euclidean preferences but, as evidenced by voter 2's indifference curve, we give that voter non-Euclidean preferences. Voter 2's ideal point is the core point, but, under condition (C5), it is a polarized equilibrium for both candidates to locate at voter 3's ideal point: none of the platforms weakly majority-preferred to $\tilde{x}_3$, in the region described by hash marks, are preferred to $\tilde{x}_3$ by either candidate. Note that the equilibrium in this example is robust, in the sense that it survives small enough variations in the gradients of the voters and candidates.

Polarized equilibria can exist even in the absence of a core point, as shown in Figure 9, when the number of voters is odd. Here, the ideal points of three voters are arranged in a triangle, and we can give the voters and candidates Euclidean preferences. It is a polarized equilibrium, under condition (C5), for the candidates to locate at voter 3's ideal point in this example, because the weakly majority-preferred platforms are those weakly preferred by voters 1 and 2. This set, being the intersection of two circles, is sufficiently kinked — so that no such platforms are preferred by either candidate — as long as 1's

Figure 8: A polarized equilibrium not at the core

Figure 9: A polarized equilibrium with no core

and 2's ideal points are far enough apart. If we moved 1's and 2's ideal points toward each other, the boundary of that set would begin to approximate the boundary of a circle, and the symmetry condition of Theorem 2 would be violated, giving one of the candidates a profitable deviation. Note that this equilibrium is also robust to small variations in preferences.

We have attempted to be as general as possible regarding the preferences of voters and candidates. We use strict quasi-concavity of voter preferences only in Lemma 1 (in the appendix) and strict quasi-concavity of candidate policy preferences only in the proof of Theorem 1. It is conceivable that weaker conditions would suffice. Differentiability is not critical for Theorem 1, but it is needed to derive the necessary conditions of Theorems 2 and 3. Regarding the strategic preferences of candidates, we have attempted to impose the weakest conditions possible. Implicit in conditions (C2)-(C4) is the idea that majority ties are broken with positive probability on each candidate's platform. This implicit assumption is critical for the proof of

Theorem 1: without it, we might have xIy and y the outcome of the election with probability one; but then we would expect candidate A to be indifferent between (x, y) and (y, y) , creating the possibility of competitive equilibria violating policy coincidence. Suppose, for example, that $d = 1$, that there is one voter with Euclidean preferences and ideal point at zero, and that the candidates' ideal points are both at -1 . In this setting, it would be a competitive equilibrium for candidate A to locate at $-1/2$ and B at $1/2$, if the voter votes for candidate B with zero probability: all of the platforms preferred by the voter to candidate A 's are less desirable, from candidate B 's perspective, than A 's platform.

An alternative specification of candidate preferences is the "mixed" model: candidate C receives a utility of $u_C(x)$ if the platform x is implemented, plus a positive utility, say β , if C is the winner of the election. In this model, condition (C2) would not hold, but the arguments of Theorem 1 could be modified to obtain the result that, in equilibrium, the candidates must adopt the same platform, say $\hat{x}$. Then it is easy to see that, when the number of voters is odd, for example, $\hat{x}$ must be a core point. If not, there is some y majority-preferred to it. That platform may be a worse policy outcome from a candidate's point of view, but every platform between $\hat{x}$ and y is also majority-preferred to $\hat{x}$. By picking such a platform close enough to $\hat{x}$, the candidate can make the disutility of the policy change less than β , the utility from winning, a contradiction. Then the symmetry of the voters' gradients follows from Plott's (1967) theorem. Clearly, driving this argument is a discontinuity in the candidates preferences, one introduced by the positive reward for winning in the mixed model. This raises the question: Are the strong necessary conditions for equilibrium existence merely an artifact of the discontinuity introduced by office motivation? Our results yield the answer: When the dimension of the policy space is at least three or the number of voters is even, those restrictive conditions are inherent in the strategic incentives of electoral competition, even with purely policy motivated candidates.

Finally, we have assumed a finite number of voters, whereas Calvert (1985) assumed a continuum of voters. This was mainly for convenience, though it also allowed us to highlight a subtlety of the n even case and to emphasize a distinction from models of office-motivated candidates. There are results suggesting that key properties of finite, n odd, electorate models carry over to continuum models. For example, Banks, Duggan, and Le Breton (1999) give a condition on the dispersion of voter ideal points, in a model of a continuous electorate, under which the second part of Lemma 1

holds: if xRy and x' is between x and y , then $x'Py$. In addition, McKelvey, Ordeshook, and Ungar (1980) extend Plott's (1967) negative result for the n odd case to a continuum of voters. We are confident that, under appropriate assumptions, the results of Theorems 1 through 3 hold in the continuous version of our model.

A Proofs of Results

Many of the arguments of this appendix will use the following standard lemma, which follows in a straightforward way from the strict quasi-concavity of the voters' utility functions.

Lemma 1 *If xRy then, for all $\alpha \in (0, 1)$, $\alpha x + (1 - \alpha)yRy$; if n is odd, moreover, then $\alpha x + (1 - \alpha)yPy$.*

We now state and prove the results of Sections 3 and 4.

Theorem 1 *Assume (C1)-(C4). If (x_A, x_B) is a competitive equilibrium, then $x_A = x_B$.*

Proof: Suppose (x_A, x_B) is a competitive equilibrium and that $x_A \neq x_B$. Without loss of generality, suppose x_ARx_B . We first assume n is odd. Note that $u_A(x_A) > u_A(x_B)$, for suppose otherwise. If $u_A(x_B) > u_A(x_A)$, then, by condition (C2), we have $(x_B, x_B) \succ_A (x_A, x_B)$, a contradiction. If $u_A(x_A) = u_A(x_B)$, then let $x' = (1/2)x_A + (1/2)x_B$. By Lemma 1, $x'Px_B$. Since u_A is strictly quasi-concave, we have $u_A(x') > u_A(x_A) = u_A(x_B)$. Then, by condition (C1), we have $(x', x_B) \succ_A (x_A, x_B)$, a contradiction. Therefore, $u_A(x_A) > u_A(x_B)$. Next, note that x_APx_B , for suppose otherwise. Then x_AIx_B . Let $\{\alpha_m\}$ be a sequence increasing to one, and define $x_m = \alpha_m x_A + (1 - \alpha_m)x_B$. By Lemma 1, x_mPx_B for all m . Then, by condition (C3), we have $(x_m, x_B) \succ_A (x_A, x_B)$ for large enough m , a contradiction. Therefore, x_APx_B . By continuity of the u_i 's, there is an open set $Y \subseteq X$ containing x_A such that, for all $x \in Y$, xPx_B . Since (x_A, x_B) is competitive, $\nabla u_A(x_A) \neq 0$. Letting $x_\epsilon = x_A + \epsilon \nabla u_A(x_A)$ for $\epsilon > 0$, and choosing ϵ close enough to zero, we have $u_A(x_\epsilon) > u_A(x_A)$ and $x_\epsilon Px_B$. Then, by condition (C1), we have $(x_\epsilon, x_B) \succ_A (x_A, x_B)$, a contradiction.

Now assume n is even. Again, $u_A(x_A) > u_A(x_B)$, for suppose otherwise. If $u_A(x_B) > u_A(x_A)$, then, as above, condition (C2) yields a contradiction. If

$u_A(x_A) = u_A(x_B)$, then, as above, let $x' = (1/2)x_A + (1/2)x_B$. If $x'Px_B$, then (C1) yields a contradiction. In the n even case, however, Lemma 1 implies only that $x'Rx_B$, so suppose $x'Ix_B$. Since the u_i 's are strictly quasi-concave, this means that $u_i(x_B) \geq u_i(x_A)$ for exactly half of the voters. Consequently, $u_i(x_A) > u_i(x_B)$ for half of the voters. Therefore, by continuity, there is an open set $Y \subseteq X$ containing x_A such that, for all $x \in Y$, xRx_B . Since (x_A, x_B) is competitive, $\nabla u_A(x_A) \neq 0$. Defining x_ϵ as above, we can choose $\epsilon > 0$ small enough that $u_A(x_\epsilon) > u_A(x_A)$ and $x_\epsilon Rx_B$. Then, by condition (C4), we have $(x_\epsilon, x_B) \succ_A (x_A, x_B)$, a contradiction. Therefore, $u_A(x_A) > u_A(x_B)$ holds. As above, x_APx_B , for suppose otherwise. Defining x_m as above, Lemma 1 now implies x_mRx_B for all m . If this majority preference holds strictly, (C3) again yields a contradiction. If x_mIx_B for any m , we repeat the preceding argument, using (C4) to establish a contradiction. Therefore, x_APx_B , and, as above, (C1) yields a final contradiction. ■

Theorem 2 *Assume n is odd, and assume (C1)-(C3). If (x_A, x_B) is a polarized equilibrium, then $x_A = x_B = \hat{x}$, where $\nabla u_k(\hat{x}) = 0$ for some voter k . For $p \in \mathbb{R}^d$,*

$$|\{i \in N \mid \exists \alpha > 0 : \nabla u_i(\hat{x}) = \alpha p\}| = |\{i \in N \mid \exists \alpha < 0 : \nabla u_i(\hat{x}) = \alpha p\}|,$$

if either $\nabla u_A(\hat{x})$ and $\nabla u_B(\hat{x})$ are linearly dependent or $\nabla u_A(\hat{x})$ and $\nabla u_B(\hat{x})$ are linearly independent and $p \in \text{cone}^\circ\{\nabla u_A(\hat{x}), \nabla u_B(\hat{x})\}$.

Proof: Consider any polarized equilibrium (x_A, x_B) . As every polarized equilibrium is competitive, we know from Theorem 1 that $x_A = x_B = \hat{x}$ for some $\hat{x} \in X$. To simplify notation, let $p_A = \nabla u_A(\hat{x})$ and $p_B = \nabla u_B(\hat{x})$, and normalize both vectors so that $\|p_A\| = \|p_B\| = 1$. We first claim that, for both candidates C , we must have $p_C \cdot y < p_C \cdot \hat{x}$ for all platforms $y \neq \hat{x}$ such that $yR\hat{x}$. Otherwise, we would have $p_C \cdot y \geq p_C \cdot \hat{x}$ for some $y \neq \hat{x}$ such that $yR\hat{x}$. It follows from Lemma 1 that $x_\alpha = \alpha\hat{x} + (1-\alpha)yPx$ for all $\alpha \in (0, 1)$. Also, $p_C \cdot x_\alpha \geq p_C \cdot \hat{x}$. Using the assumption that $\hat{x}$ is interior to X , we take α close enough to one that x_α is also interior to X . Since the u_i 's are continuous, there is an open set $Y \subseteq X$ containing x_α such that, for all $z \in Y$, zPx . Defining $z_\beta = x_\alpha + \beta p_C$, we take β small enough that $z_\beta \in Y$, and therefore $z_\beta Px$. By construction,

$$p_C \cdot (z_\beta - \hat{x}) = p_C \cdot (x_\alpha - \hat{x}) + \beta p_C \cdot p_C > 0.$$

Finally, define $w_\gamma = \gamma\hat{x} + (1 - \gamma)z_\beta$. Again using Lemma 1, $w_\gamma P\hat{x}$ for all $\gamma \in (0, 1)$. Since $p_C \cdot (w_\gamma - \hat{x}) > 0$, we may take γ close enough to one that $u_C(w_\gamma) > u_C(\hat{x})$. But then, by condition (C1), we have $(w_\gamma, \hat{x}) \succ_A (\hat{x}, \hat{x})$ or $(\hat{x}, w_\gamma) \succ_B (\hat{x}, \hat{x})$ (depending on the identity of C), a contradiction. This establishes the claim.

If the gradients of the candidates are linearly dependent at the equilibrium platform $\hat{x}$, then, since they are non-zero and do not point in the same direction, it follows that the gradients point in opposite directions: $\alpha p_A = p_B$ for some $\alpha < 0$. Take any $y \neq \hat{x}$ such that yRx . From the above claim, $p_A \cdot y < p_A \cdot \hat{x}$ and $p_B \cdot y < p_B \cdot \hat{x}$. But, since the gradients of the candidates point in opposite directions, the latter yields $p_A \cdot y > p_A \cdot \hat{x}$, a contradiction. Therefore, $\hat{x}Py$ for all $y \neq \hat{x}$, which implies $\hat{x}$ is a core point. Then Plott's (1967) theorem implies that $\hat{x}$ is the ideal point of at least one voter and that the symmetry condition holds for all $p \in \mathbb{R}^d$.

Now consider the case in which the candidates' gradients are linearly independent, and suppose that, for all voters i , $\nabla u_i(\hat{x}) \neq 0$. To deduce a contradiction, we will first find a vector $q \in \mathbb{R}^d$ such that $p_A \cdot q > 0$ and $p_B \cdot q < 0$. Construct q as follows. Let $p \in \text{cone}^\circ\{p_A, p_B\}$ be any vector satisfying $p = \alpha p_A + \beta p_B$ for some $\alpha, \beta > 0$. Since p_A and p_B are linearly independent, we have $p_A \cdot p < \|p\|$. Let q be p_A minus the projection of p_A onto the one-dimensional subspace spanned by p , i.e.,

$$q = p_A - \frac{(p_A \cdot p)}{(p \cdot p)} p.$$

Then, since $p_A \cdot p_A = 1$ and $(p_A \cdot p)/\|p\| < 1$, we have $p_A \cdot q > 0$. Furthermore, $q \cdot p = 0$, implying $p_B \cdot q = -(\alpha/\beta)p_A \cdot q < 0$. This gives us a vector q with the desired properties. In fact, there is an open set Q containing q such that, for all $s \in Q$, $p_A \cdot s > 0$ and $p_B \cdot s < 0$. Because N is finite and $\nabla u_i(\hat{x}) \neq 0$ for all voters i , we may choose $r \in Q$ so that $r \cdot \nabla u_i(\hat{x}) \neq 0$ for all i . Therefore, since the voters are odd in number, either

$$\{i \in N \mid r \cdot \nabla u_i(\hat{x}) > 0\} \quad \text{or} \quad \{i \in N \mid r \cdot \nabla u_i(\hat{x}) < 0\}$$

contains a majority of voters. Suppose, without loss of generality, that this is true for the first group of voters, and define $x_\epsilon = \hat{x} + \epsilon r$ for $\epsilon > 0$. Since $\hat{x}$ is interior to X , we may choose ϵ small enough that $x_\epsilon \in X$. Furthermore, since $\nabla u_i(\hat{x}) \cdot (x_\epsilon - \hat{x}) > 0$ for a majority of voters, $x_\epsilon P\hat{x}$ for ϵ close enough to zero. And, since $p_A \cdot (x_\epsilon - \hat{x}) = \epsilon p_A \cdot r > 0$, we have $u_A(x_\epsilon) > u_A(\hat{x})$ for ϵ

close enough to zero. But then, by condition (C1), we have $(x_\epsilon, \hat{x}) \succ_A (\hat{x}, \hat{x})$, a contradiction. Therefore, $\nabla u_k(\hat{x}) = 0$ for some voter k .

Now take any $p \in \text{cone}^\circ\{p_A, p_B\}$, and suppose the symmetry condition of the theorem is violated. We will show that one of the candidates has a profitable deviation, a contradiction. Let $\sigma = 1$ if

$$|\{i \in N \mid \exists \alpha > 0 : \nabla u_i(\hat{x}) = \alpha p\}| > |\{i \in N \mid \exists \alpha < 0 : \nabla u_i(\hat{x}) = \alpha p\}|,$$

and let $\sigma = -1$ if the opposite inequality holds. As above, pick $q \in \mathbb{R}^d$ such that $p \cdot q = 0$, $p_A \cdot q > 0$, and $p_B \cdot q < 0$. Let Q be an open set on which these strict inequalities hold, and let $Q' = \{s \in Q \mid p \cdot s = 0\}$ be the elements of that set orthogonal to p . For $r \in Q'$, let $O(r) = \{s \in \mathbb{R}^d \mid s \cdot r = 0\}$ denote the subspace orthogonal to r . We claim that $\bigcap_{r \in Q'} O(r) = \text{span}\{p\}$. To see this, let $\{b_1, \dots, b_{d-1}\}$ be a basis for the $(d-1)$ -dimensional subspace orthogonal to p , and take $r \in Q'$ and $\epsilon > 0$ such that $\{r + \epsilon b_1, \dots, r + \epsilon b_{d-1}\}$ is linearly independent and contained in Q' . By linear independence, the dimension of

$$\bigcap_{h=1}^{d-1} O(r + \epsilon b_h)$$

is one. Of course, $p \in O(r)$ for all $r \in Q'$, establishing the claim.

Then, since N is finite and k is the only voter with ideal point $\hat{x}$, choose $r \in Q'$ so that $r \cdot \nabla u_i(\hat{x}) = 0$ if and only if $i = k$ or, for some $\alpha \neq 0$, $\nabla u_i(\hat{x}) = \alpha p$. Partition $N \setminus \{k\}$ into four sets,

$$\begin{aligned} I &= \{i \in N \mid r \cdot \nabla u_i(\hat{x}) > 0\} \\ J &= \{i \in N \mid r \cdot \nabla u_i(\hat{x}) < 0\} \\ K &= \{i \in N \mid \exists \alpha > 0 : \nabla u_i(\hat{x}) = \sigma \alpha p\} \\ L &= \{i \in N \mid \exists \alpha < 0 : \nabla u_i(\hat{x}) = \sigma \alpha p\}, \end{aligned}$$

and note that $|K| > |L|$. Without loss of generality, suppose $|I| \geq |J|$. Since $N \setminus \{k\}$ contains $n-1$ voters, we have $|K| + |I| > (n-1)/2$, and this implies $|K| + |I| \geq (n+1)/2 > n/2$. We will use r to construct a profitable deviation for candidate A . (If the inequality $|I| < |J|$ held instead, we would use $-r$ to construct a profitable deviation for B .) Let $x_\delta = \hat{x} + \delta r$ for $\delta > 0$. Then $\nabla u_i(\hat{x}) \cdot (x_\delta - \hat{x}) = \delta \nabla u_i(\hat{x}) \cdot r > 0$ for all $i \in I$, and $p_A \cdot (x_\delta - \hat{x}) = \delta \nabla p_A \cdot r > 0$. Choose δ close enough to zero that x_δ is interior

to X . Define $x_\epsilon = x_\delta + \epsilon \sigma p$ for $\epsilon > 0$, and choose ϵ close enough to zero that, for all $i \in I$, $\nabla u_i(\hat{x}) \cdot (x_\epsilon - \hat{x}) > 0$; and small enough that $p_A \cdot (x_\epsilon - \hat{x}) > 0$. Note that, since $\nabla u_i(\hat{x}) \cdot r = 0$ for all $i \in K$, we have

$$\begin{aligned}\nabla u_i(\hat{x}) \cdot (x_\epsilon - \hat{x}) &= \delta \nabla u_i(\hat{x}) \cdot r + \epsilon \nabla u_i(\hat{x}) \cdot \sigma p \\ &> 0\end{aligned}$$

for all $i \in K$. Picking ϵ close enough to zero, we have $x_\epsilon \in X$ and, for all $i \in I \cup K$, $u_i(x_\epsilon) > u_i(\hat{x})$, which implies $x_\epsilon P \hat{x}$. Furthermore, $u_A(x_\epsilon) > u_A(\hat{x})$. But then condition (C1) implies that $(x_\epsilon, \hat{x}) \succ_A (\hat{x}, \hat{x})$, a contradiction. Therefore, the symmetry condition of the theorem must hold. ■

Theorem 3 *Assume n is odd, and assume (C1)-(C3). If (x_A, x_B) is a polarized equilibrium, then $x_A = x_B = \hat{x}$, where $\nabla u_k(\hat{x}) = 0$ for some voter k . Moreover, for every $p \in \mathbb{R}^d$ such that $p \notin \text{span}\{\nabla u_A(\hat{x}), \nabla u_B(\hat{x})\}$,*

$$|\{i \in N \mid \exists \alpha > 0 : \nabla u_i(\hat{x}) = \alpha p\}| = |\{i \in N \mid \exists \alpha < 0 : \nabla u_i(\hat{x}) = \alpha p\}|.$$

Proof: Let (x_A, x_B) be a polarized equilibrium. From Theorem 2, it follows that $x_A = x_B = \hat{x}$, where $\nabla u_k(\hat{x}) = 0$ for some voter k . As shown in the proof of Theorem 2, if the gradients of the candidates are linearly dependent, then $\hat{x}$ is a core point, and the symmetry condition of the theorem is satisfied. We assume, then, that their gradients are linearly independent. As above, let $p_A = \nabla u_A(\hat{x})$ and $p_B = \nabla u_B(\hat{x})$ and normalize so that $\|p_A\| = \|p_B\| = 1$. Moreover, for every voter i , let $p_i = \nabla u_i(\hat{x})$. Given $q, r \in \mathbb{R}^d$, let $S(q, r) = \text{span}\{q, r\}$ denote the subspace spanned by q and r . We will take q and r to be linearly independent, implying that $S(q, r)$ is a two-dimensional subspace, i.e., a plane. Given $p, q, r \in \mathbb{R}^d$, let

$$p(q, r) = \text{proj}_{S(q, r)} p$$

denote the projection of p onto the span of $\{q, r\}$. Thus, $p_C(q, r)$ would be the projection of candidate C 's gradient onto that plane. Given $p \in \mathbb{R}^d$, let

$$O(p) = \{q \in \mathbb{R}^d \mid q \cdot p = 0\}$$

denote the subspace orthogonal to p . Given $p \in \mathbb{R}^d$, let

$$S(p) = \text{span}\{\text{proj}_{O(p)} p_A, \text{proj}_{O(p)} p_B\}$$

denote the two-dimensional subspace spanned by the projections of the candidates' gradients onto the space orthogonal to p . Given $p, q \in \mathbb{R}^d$, let

$$q(p) = \text{proj}_{S(p)} q$$

denote the projection of q onto that plane. Note that, since $\text{proj}_{O(p)} p_C \in S(p)$ and $S(p) \subseteq O(p)$, we have

$$p_C(p) = \text{proj}_{S(p)} p_C = \text{proj}_{O(p)} p_C,$$

so $p_C(p)$ is just the gradient of candidate C projected onto the subspace orthogonal to p . That, in turn, implies $S(p_A(p), p_B(p)) = S(p)$. Finally, note the further implication that $q(p) = q(p_A(p), p_B(p))$.

Let $q, r \in \mathbb{R}^d$ be vectors such that the gradients of the candidates, projected onto the plane $S(q, r)$, point in different directions, i.e., there do not exist $\alpha, \beta \geq 0$, at least one non-zero, such that $\alpha p_A(q, r) = \beta p_B(q, r)$. Thus, if we restrict the candidates' platforms to the two-dimensional space $\hat{x} + S(q, r)$, then the pair $(\hat{x}, \hat{x})$ is a polarized equilibrium of the restricted game. Take any $p \in \text{cone}^\circ\{p_A(q, r), p_B(q, r)\}$ in the open cone generated by the candidates' projected gradients, so that the antecedent conditions of Theorem 2 hold in the restricted game. We claim that

$$\begin{aligned} & |\{i \in N \mid \exists \alpha > 0 : p_i(q, r) = \alpha p(q, r)\}| \\ &= |\{i \in N \mid \exists \alpha < 0 : p_i(q, r) = \alpha p(q, r)\}|. \end{aligned}$$

If not, then, by Theorem 2, one of the candidates has a profitable deviation in the restricted game, and therefore the candidate has a profitable deviation in the original game, a contradiction. This establishes the claim.

To prove the theorem, take any $p \notin \text{span}\{p_A, p_B\}$, normalize so $\|p\| = 1$, let

$$\begin{aligned} I &= \{i \in N \mid \exists \alpha > 0 : \nabla u_i(\hat{x}) = \alpha p\} \\ J &= \{i \in N \mid \exists \alpha < 0 : \nabla u_i(\hat{x}) = \alpha p\}, \end{aligned}$$

and suppose that $|I| \neq |J|$. Without loss of generality, suppose $|I| > |J|$. In light of the above claim, a contradiction is proved if we find vectors q and r satisfying three conditions:

- (1) there do not exist $\alpha, \beta \geq 0$, at least one non-zero, such that $\alpha p_A(q, r) = \beta p_B(q, r)$;

- (2) $p(q, r) \in \text{cone}^\circ\{p_A(q, r), p_B(q, r)\}$;
 (3) the symmetry condition of Theorem 2 in the game restricted to $\hat{x} + S(q, r)$ is violated, specifically,

$$\begin{aligned} I &= \{i \in N \mid \exists \alpha > 0 : p_i(q, r) = \alpha p(q, r)\} \\ J &= \{i \in N \mid \exists \alpha < 0 : p_i(q, r) = \alpha p(q, r)\}. \end{aligned}$$

We first consider the possibility of setting $q = p_A(p)$ and $r = p_B(p)$. As noted above, we would then have $p_A(q, r) = p_A(p)$ and $p_B(q, r) = p_B(p)$, so condition (1) would be satisfied if $p_A(p)$ and $p_B(p)$ were linearly independent. To show this, note that there exist unique, non-zero α and β such that

$$p_A = p_A(p) + \alpha p \quad \text{and} \quad p_B = p_B(p) + \beta p.$$

If $p_A(p)$ and $p_B(p)$ are linearly dependent, then there exist γ and δ , at least one non-zero, such that $\gamma p_A(p) + \delta p_B(p) = 0$. But then

$$\gamma p_A + \delta p_B = (\alpha\gamma + \beta\delta)p,$$

which implies

$$p = \left(\frac{\gamma}{\alpha\gamma + \beta\delta} \right) p_A + \left(\frac{\delta}{\alpha\gamma + \beta\delta} \right) p_B,$$

contradicting $p \notin \text{span}\{p_A, p_B\}$. Therefore, the projected gradients of the candidates are linearly independent, as claimed.

We cannot simply set $q = p_A(p)$ and $r = p_B(p)$, however, because then we would have

$$p(q, r) = p(p_A(p), p_B(p)) = 0,$$

violating condition (2). Next, we establish the existence of a perturbation, s , of p such that conditions (1) and (2) are both satisfied by $q = p_A(s)$ and $r = p_B(s)$. As noted above, $p(s) = p(p_A(s), p_B(s))$ and $p_C(s) = p_C(p_A(s), p_B(s))$ for each candidate, so condition (2) can be written as $p(s) \in \text{cone}^\circ\{p_A(s), p_B(s)\}$. To construct the perturbation, let $s_\epsilon = p - (\epsilon/2)(p_A + p_B)$ for $\epsilon > 0$. Note that, by linearity of the projection mapping and $s_\epsilon(s_\epsilon) = 0$,

$$\begin{aligned} p(s_\epsilon) &= (s_\epsilon + (\epsilon/2)p_A + (\epsilon/2)p_B)(s_\epsilon) \\ &= (\epsilon/2)p_A(s_\epsilon) + (\epsilon/2)p_B(s_\epsilon). \end{aligned}$$

Thus, $p(s_\epsilon) \in \text{cone}^\circ\{p_A(s_\epsilon), p_B(s_\epsilon)\}$. Taking ϵ close enough to zero that $p_A(s_\epsilon)$ and $p_B(s_\epsilon)$ are linearly independent, we set $s = s_\epsilon$ for the desired perturbation.

We now wish to find perturbations, q and r , of $p_A(s)$ and $p_B(s)$ that satisfy condition (3) as well as (1) and (2). Let voter j satisfy $p_j(s) = \alpha p(s)$ for some $\alpha < 0$ but $p_j \neq \alpha p$. That is, although the voter's gradient appears to point in the $-p$ direction when projected, the voter is not a member of J . Note the immediate implication that p_j and p are linearly independent. We will find arbitrarily close vectors v and w such that $p_j(v, w) = \alpha' p(v, w)$ for no $\alpha' < 0$. Note that

$$\begin{aligned} p_j \cdot p_A(s) &= (p_j - p_j(s)) \cdot p_A(s) + p_j(s) \cdot p_A(s) \\ &= p_j(s) \cdot p_A(s) \\ &= \alpha p(s) \cdot p_A(s) \\ &= \alpha(p(s) - p) \cdot p_A(s) + \alpha p \cdot p_A(s) \\ &= \alpha p \cdot p_A(s), \end{aligned}$$

where the second equality follows from $(p_j - p_j(s)) \cdot p_A(s) = 0$ and the fourth equality from $(p(s) - p) \cdot p_A(s) = 0$. Similarly, $p_j \cdot p_B(s) = \alpha p \cdot p_B(s)$. These equalities imply

$$\frac{p_j \cdot p_A(s)}{p_j \cdot p_B(s)} = \frac{p \cdot p_A(s)}{p \cdot p_B(s)}.$$

Since p_j and p are linearly independent, there exists $t \in \mathbb{R}^d$ such that $p_j \cdot t > 0$ and $p \cdot t < 0$. Define $v_\epsilon = p_A(s) + \epsilon t$ and $w_\epsilon = p_B(s) - \epsilon t$ for $\epsilon > 0$, and note that

$$\frac{p_j \cdot v_\epsilon}{p_j \cdot w_\epsilon} > \frac{p \cdot v_\epsilon}{p \cdot w_\epsilon}.$$

Thus, $p_j(v_\epsilon, w_\epsilon) = \alpha' p(v_\epsilon, w_\epsilon)$ for no $\alpha' < 0$. That is, the gradient of voter j , projected onto the plane spanned by v_ϵ and w_ϵ , no longer appears to point in the $-p$ direction. Since conditions (1) and (2) hold on open sets around $p_A(s)$ and $p_B(s)$, we can choose ϵ small enough that (1) and (2) hold for v_ϵ and w_ϵ . Since N is finite, we can perturb v_ϵ and w_ϵ a finite number of times, if needed, so that the only voters whose projected gradients point in the $-p(v_\epsilon, w_\epsilon)$ direction are the members of J . By a similar argument, we can perturb v_ϵ and w_ϵ so that the only voters whose projected gradients point in the $p(v_\epsilon, w_\epsilon)$ direction are the members of I , fulfilling condition (3). ■

Theorem 4 Assume n is even, and assume (C1)-(C4). There does not exist a polarized equilibrium.

Proof: To prove the theorem, consider any polarized equilibrium (x_A, x_B) . By Theorem 1, the candidates must locate at the same platform, say $\hat{x} = x_A = x_B$. We claim that $\nabla u_k(\hat{x}) = 0$ for some voter k , for suppose not. As in the proof of Theorem 2, let $r \in \mathbb{R}^d$ be such that $p_A \cdot r > 0 > p_B \cdot r$ and such that $r \cdot \nabla u_i(\hat{x}) = 0$ for no voter i . Then either

$$\{i \in N \mid r \cdot \nabla u_i(\hat{x}) > 0\} \quad \text{or} \quad \{i \in N \mid r \cdot \nabla u_i(\hat{x}) < 0\}$$

contains at least half of the voters. Suppose, without loss of generality, that this is true for the first group of voters, and define $x_\epsilon = \hat{x} + \epsilon r$ for $\epsilon > 0$. Since $\hat{x}$ is interior to X , we may choose ϵ small enough that $x_\epsilon \in X$. Furthermore, since $\nabla u_i(\hat{x}) > 0$ for at least half of the voters, $x_\epsilon R \hat{x}$ for ϵ close enough to zero. And, since $p_A \cdot (x_\epsilon - \hat{x}) = \epsilon p_A \cdot r > 0$, we have $u_A(x_\epsilon) > u_A(\hat{x})$ for ϵ close enough to zero. But then, by condition (C4), we have $(x_\epsilon, \hat{x}) \succ_A (\hat{x}, \hat{x})$, a contradiction. Therefore, $\nabla u_k(\hat{x}) = 0$ for some voter k .

Now consider the model with k removed from the set of voters, i.e., let the set of voters be $N' = N \setminus \{k\}$, now odd in number. Because we assumed the voters in N had distinct ideal points, there is no voter with ideal point at $\hat{x}$ in the modified model (with k removed). Following the proof of Theorem 2, one of the candidates, say A , can move to some platform x' such that $u_A(x') > u_A(\hat{x})$ and $x' P' \hat{x}$, where P' represents the strict majority preference relation in the modified model. That is, a majority of voters in N' strictly prefer x' to $\hat{x}$. Returning to the original model, that means that at least half of the voters in N strictly prefer x' to $\hat{x}$. Therefore, we have $u_A(x') > u_A(\hat{x})$ and $x' R \hat{x}$. But then, by condition (C4), we have $(x', \hat{x}) \succ_A (\hat{x}, \hat{x})$, a contradiction. ■

Proposition 1 Assume (C1)-(C4). Assume that, if candidate A has an ideal point $\tilde{x}_A$, then there exists a platform $x \in X$ such that $x P \tilde{x}_A$ and $u_B(x) > u_B(\tilde{x}_A)$; likewise, if candidate B has an ideal point $\tilde{x}_B$, then there exists a platform $y \in X$ such that $y P \tilde{x}_B$ and $u_A(y) > u_A(\tilde{x}_B)$. If (x_A, x_B) is an interior equilibrium, then either it is competitive or n is even and $x_A = \tilde{x}_A$ and $x_B = \tilde{x}_B$.

Proof: It is sufficient to show that the only interior equilibria in which one candidate, say A , adopts his/her ideal point occur when n is even and B

also adopts his/her ideal point. We first assume n is odd. Suppose $(\tilde{x}_A, x_B)$ is an interior equilibrium. There are three cases to check. First, $\tilde{x}_A P x_B$. Letting $x P \tilde{x}_A$ and $u_B(x) > u_B(\tilde{x}_A)$, condition (C1) implies that $(\tilde{x}_A, x) \succ_B (\tilde{x}_A, x_B)$, a contradiction. Second, $\tilde{x}_A I x_B$. As in the proof of Theorem 1, $u_B(x_B) > u_B(\tilde{x}_A)$. Then, by condition (C3) and Lemma 1, B can gain by moving toward $\tilde{x}_A$ a small amount, a contradiction. Third, $x_B P \tilde{x}_A$. By continuity of the u_i 's, there is an open set of platforms containing x_B that are majority-preferred to $\tilde{x}_A$. Then, by condition (C1) and our assumption that $(\tilde{x}_A, x_B)$ is an equilibrium, we have $\nabla u_B(x_B) = 0$, i.e., $x_B = \tilde{x}_B$. Then, as B could in the first case, candidate A can gain by moving to a platform y such that $y P x_B$ and $u_A(y) > u_A(x_B)$, a contradiction. If n is even, then we need modify the above argument only in the second case. Then $\tilde{x}_A I x_B$, and Lemma 1 implies that moving slightly toward $\tilde{x}_A$ leads to a platform majority-preferred or -indifferent to $\tilde{x}_A$. If the latter, then, as in the proof of Theorem 1, there is an open set of platforms containing x_B majority-indifferent to $\tilde{x}_A$. If $\nabla u_B(x_B) \neq 0$, then (C4) yields a contradiction. Otherwise, $x_B = \tilde{x}_B$, and both candidates are at their ideal points. If the former, then the argument in Theorem 1 yields a contradiction. ■

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