IMAGE ANALYSIS OF CURVED GRADIENT INDEX RODS

by

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CURRICULUM VITAE

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ABSTRACT

A method of ray tracing through curved gradient index imaging rods is developed.

A study of the imaging ability of a radial gradient index relay rod, with end-to-end imaging, bent along some simple curves is performed. The study includes an analysis of changes in the first order properties, and the transverse aberrations in the image plane. Approximations of the first, second, and third order transverse aberration coefficients are made.

In order to improve the imaging ability of the rod, modifications of the index profile are considered for the case of a rod bent along a parabolic curve.

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Chapter I

INTRODUCTION

Considerable progress has been made in the field of Gradient Index Optics since the early investigations of Maxwell [1.1] and Wood [1.2]. This progress is evident through the development of unique gradient index imaging systems with performances which are unattainable by conventional homogeneous systems. Numerous applications of gradient index (GRIN) components in optical designs have been reported. Included among these systems are the Schmidt corrector plate [1.3], fiber and rod couplers [1.4], photographic objectives [1.5], and eyepiece designs [1.6].

Recent interest in gradient index was sparked by the introduction of the first commercially available GRIN component. The cylindrical gradient index imaging rod made in Japan by Nippon Sheet Glass is referred to as a Selfoc[®] rod [1.7]. The Selfoc[®] is being used in the photocopy industry and in medical optics applications. The photocopy industry uses the rod's short conjugate distance to achieve the compactness required of a desktop copier. In the field of medical optics, the small diameter and imaging ability of

the rod is being used in an endoscope, known as the Needlescope® [1.8].

To achieve its present level of technology, research in gradient index optics has developed in many directions. Three of the major subdivisions are the design and analysis of gradient index systems, the manufacturing of gradient index materials, and techniques for measuring the materials' optical properties.

Development of the aberration theory [1.9, 1.10] and ray tracing algorithms [1.11-1.17] for inhomogeneous media has allowed designers to assess the advantages provided by GRIN elements, and to determine the optimum index profiles (the functional forms of the index variation in the media). Some of the ray trace algorithms are solutions of the ray path for specific index gradients. Marchand [1.11, 1.12] has developed separate methods for ray tracing in axial, spherical, and radial gradients. Paxton and Streifer [1.13, 1.14] have derived an analytic solution for the ray path in a cylindrical GRIN system.

A variety of algorithms are available for general raytracing in inhomogeneous media. Moore's [1.15] algorithm is based on a polynomial expansion of the ray path. The index of refraction is assumed to be rotationally symmetric about the optical axis and is expressed as a power series.

Incremental methods have been developed by Montagnino [1.16] and Sharma, et. al. [1.17]. Montagnino's method is a point by point extrapolation using a Taylor series expansion of the ray path over a small region. Sharma transforms the ray equation into a convenient form and solves the resulting equation using a numerical technique.

Currently the fabrication techniques being pursued in order to make these index profiles in optical materials include ion exchange, chemical vapor deposition, ion implantation, crystal growth, and photopolymerization of plastics [1.3]. Fabrication methods are not currently available to meet the demands of the designers. This is the primary reason for the limited availability of GRIN components.

The capability to measure and test the fabricated material is the third significant area of research. A method which is very adaptable for GRIN material testing is A.C. (heterodyne) interferometry. This technique is the basis for many interferometric systems for measuring the index profile [1.19], dispersion properties [1.20], and thermal properties [1.21] of GRIN materials. Another method capable of making measurements of the absolute index profile uses an ellipsometry technique [1.18].

The commercial availability of the Selfoc[®] rod has provoked an increasing interest in the use of GRIN rods as optical image relays. The transmission of spatial information can be performed using a variety of relay systems. Among the conventional optical relays are bundles of fibers and standard lens systems. Each of these systems has inherent disadvantages which typify the problems associated with the transfer of spatial information. A fiber bundle consists of a number of graded or step-index fibers. In the case of the step-index fiber bundle, each individual fiber is non-imaging, therefore, the information transferred is limited to one pixel per fiber. This limits the resolution of the image and the amount of information that can be transferred. The resolution of the graded-index fiber, although the individual fibers are imaging, is limited due to the small diameter of each core.

A standard lens system relay usually consists of a number of small elements and, therefore, has the disadvantages associated with small diameter optics. The physical size of the elements imposes a restriction on the amount of information that can be transferred. Alignment difficulties which are inherent to small components can also cause problems. For long relays, a large number of elements is needed, therefore light loss due to the surface reflections can be significant.

The development of gradient index rods has provided an alternative to the conventional image relay system. Many of the problems of light loss, alignment, and limited resolution are eliminated by the imaging capability of the GRIN rods. The resolution is limited by the aberration correction of the rod or array of rods. Light loss and alignment problems are reduced since there is only a single rigid component as the primary element of the relay.

The improved image quality of the GRIN rod has opened the way for further application of image relays. One application is in long relay systems. A complication that arises in this use of the GRIN rod is curvature along the axis of the rod. The possibility, and inevitability, of curvature of the rod increases with the length of the rod. A complete analysis of such a situation is one of the topics of this thesis.

Studies in the area of gradient index optical systems with bent axes have predominantly occurred in the field of telecommunications using graded-index fibers. A number of authors have reported on this subject [1.22, 1.23, 1.27]. One study using an aberration theory approach, which is appropriate for gradient index fibers as well as the larger diameter rods, has been performed by Rohra and Thyagarajan [1.24]. Starting from Hamilton's equations, explicit

expressions for the second and third order aberration coefficients of a fiber bent along the arc of a circle have been obtained. The index profile considered is a function of the transverse distance from the curved rod's physical axis and the distance along this axis. It was shown that the second and third order aberrations critically depend on the gradient index parameters of the fiber, and on the bending.

The bending of imaging GRIN rods has not been examined in detail since a large diameter glass rod is considered a rigid structure. Research in new optical materials for gradient index has recently resulted in the fabrication of flexible, light-focusing plastic rods [1.25] (LFR). Ohtsuka et. al. [1.26] have characterized the image transferred through a flexible plastic rod curved along the arc of a circle. It was noted that the vignetting of the image is proportional to the curvature of the bend, and that an image shift occurs with bending.

The use of long glass imaging GRIN rods and the development of flexible, light-focusing, plastic rods indicates a need for the complete analysis of a bent, or curved, image relay. The development of such an analysis is outlined in this thesis. A number of questions are considered. What method of ray tracing is the most

suitable? How should the index profile and the coordinate system be represented to maintain a general form capable of handling a variety of gradient index systems? What assumptions concerning the material properties of glass rods are reasonable when bending the rods?

Some optical relay systems are analyzed for the effects on image quality caused by bending. Finally, modifications to the index profile to correct any degradation of image quality are considered.

Chapter II

THEORY

2.1 INTRODUCTION

An imaging gradient index (GRIN) rod is shown in Figure 2.1. It should be observed that the notation of this manuscript uses x as the optical axis in the right-handed coordinate system (Y, Z, x) . The index variation of the GRIN rod in Figure 2.1 is a function of the radial distance from the axis. The a and b -rays demonstrate the characteristic sinusoidal ray path. The period is determined by the index profile, and the number of internal images in a given rod depends upon the length of one period and the total length of the rod.

The ray path and image quality of a curved GRIN rod system are not completely predictable from knowledge of an equivalent straight rod system. Equivalent rods, whether straight or curved, are GRIN rods which have the same index variation relative to the physical axis of the rod. The curved rod does not possess an optical axis of rotational symmetry; therefore, even-order aberrations are present. The evaluation of the second and third order aberrations for

Imaging Grin Rod: Plus-One Magnification

Figure 2.1

the particular case of a rod bent along the arc of a circle has been performed [2.1]. To completely analyze the aberrations of a generally curved rod with a maximum of accuracy, a real ray analysis is necessary. This requires a ray trace in which an axis of rotational symmetry is not assumed. Such a ray trace was developed and implemented in Fortran on the IBM-3032 at the University of Rochester. Its development is outlined in the following sections and examples of its use are found in Chapter 3.

2.2 RAY TRACING ALGORITHM

An incremental approach was chosen as the ray tracing method for the inhomogeneous media. The technique selected was developed by Sharma et. al. [2.2] and is one in which the ray equation is transformed into a form convenient for direct numerical integration.

The ray path in a gradient index medium is given by

$$\frac{d}{ds} \left[n(\vec{r}) \frac{d\vec{r}}{ds} \right] = \vec{\nabla} n(\vec{r}) \quad (2.1)$$

where

$$\vec{r} = y\hat{j} + z\hat{k} + x\hat{i} \quad ,$$

$\vec{r}$ is the position vector of a point on the ray, $n(\vec{r})$ is the functional form of the refractive index, and ds is an element of the arc length along the ray. The transformation of the ray equation is performed by making the substitution

$$t = \int \frac{ds}{n} \quad (2.2a)$$

therefore,

$$dt = \frac{ds}{n} . \quad (2.2b)$$

The ray Equation (2.1) can be rewritten as

$$\frac{d^2 \vec{r}}{dt^2} = n \vec{\nabla} n , \quad (2.3a)$$

or

$$\frac{d^2 \vec{r}}{dt^2} = \frac{1}{2} \vec{\nabla} n^2 . \quad (2.3b)$$

Equation (2.3) can be integrated numerically to determine the ray path using the Runge-Kutta algorithm. The form of Equation (2.3), a or b, that is used depends upon whether

the representation of the refractive index is in terms of, n or n^2 . The three components of $\vec{r}$ are determined using three simultaneous equations. To write the integration algorithm in a compact notation it is advantageous to define an optical ray vector, $\vec{T}$, and to adopt a matrix notation.

The optical ray vector, $\vec{T}$, is defined as

$$\vec{T} = \frac{d\vec{r}}{dt} = n \frac{d\vec{r}}{ds} \quad (2.4)$$

and since

$$\begin{aligned} \frac{d}{ds} (\vec{r}) &= \frac{d}{ds} (y\hat{j} + z\hat{k} + x\hat{i}) \\ &= \frac{dy}{ds} \hat{j} + \frac{dz}{ds} \hat{k} + \frac{dx}{ds} \hat{i} \\ &= \hat{j} \cos\beta + \hat{k} \cos\gamma + \hat{i} \cos\alpha, \end{aligned} \quad (2.5)$$

$\vec{T}$ becomes

$$\vec{T} = (n \cos\beta)\hat{j} + (n \cos\gamma)\hat{k} + (n \cos\alpha)\hat{i} \quad (2.6)$$

The three terms of $\vec{T}$ are the product of the index and the optical direction cosines.

The following definitions are used in the matrix notation:

$$R \equiv \begin{bmatrix} Y \\ Z \\ x \end{bmatrix} \quad (2.7)$$

$$\vec{T} \equiv \begin{bmatrix} T_Y \\ T_Z \\ T_x \end{bmatrix} \equiv n \begin{bmatrix} dY/ds \\ dZ/ds \\ dx/ds \end{bmatrix} \equiv n \begin{bmatrix} \cos\beta \\ \cos\gamma \\ \cos\alpha \end{bmatrix} \quad (2.8)$$

$$D \equiv n \begin{bmatrix} dn/dY \\ dn/dZ \\ dn/dx \end{bmatrix} \equiv \frac{1}{2} \begin{bmatrix} dn^2/dY \\ dn^2/dZ \\ dn^2/dx \end{bmatrix} . \quad (2.9)$$

Using the matrix notation the ray Equation (2.3) can be rewritten as

$$\frac{d^2 R}{dt^2} = D(R) . \quad (2.10)$$

The Runge-Kutta algorithm is used to solve Equation (2.10) and is outlined by the following equations:

$$\begin{aligned} R_{n+1} &= \Delta t [T_n + 1/6(A + 2B)] \\ T_{n+1} &= 1/6(A + 4B + C) , \end{aligned} \quad (2.11)$$

where

$$\begin{aligned}
 A &= \Delta t D(R_n), \\
 B &= \Delta t D(R_n + \frac{\Delta t}{2} T_n + 1/8 \Delta t A), \\
 C &= \Delta t D(R_n + \Delta t T_n + \frac{1}{2} \Delta t B),
 \end{aligned} \tag{2.12}$$

and Δt is the extrapolation distance, the incremental value of t . Given the initial position, $R_0 (Y_0, Z_0, x_0)$ and direction, $T_0 (n_0, \beta_0, \gamma_0, \alpha_0)$, of a ray, its path, $(R_1, T_1), (R_2, T_2), (R_3, T_3), \dots (R_n, T_n)$, can be determined if the value of the index and its gradient are known at each position. The accuracy of the ray trace depends on the value of Δt which is supplied by the user. For steeper gradients a smaller value of Δt is necessary to maintain a given accuracy.

2.3 REPRESENTATION OF THE BENT ROD

In order to implement the ray trace developed by Sharma et. al. [2.2], the value of the index and its gradient must be known at each incremental step in the media. It is not necessary to have a functional form of the index gradient, ∇n or ∇n^2 , since the gradient can be calculated numerically if the index is known for a set of closely spaced points.

To determine the index at any point within a bent rod requires a means of describing the general bend of a rod and the development of a general index representation along this bend.

The bend of the rod is described by the curve in the (Y, Z, x) coordinate system along which the physical axis of the rod lies (see Figure 2.2). The front face of the rod is assumed to lie in the YZ plane at $x = 0$. The curve of the physical axis is written as two fifth-order parametric equations in x

$$\begin{aligned} Y(x) &= a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5, \\ Z(x) &= b_1x + b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5. \end{aligned} \tag{2.13}$$

Thus, the location of the bent rod is fully described by the values of x.

The development of a general index representation along this curve is achieved by first describing an equivalent straight rod with its symmetric or asymmetric index distribution. A series of manipulations of the index profile are performed to simulate the physical bending of the straight rod to match the desired curved rod. The general index distribution thereby lies along the physical axis of the curved rod.

Axis Representation of the Curved GRIN Rod

Figure 2.2

The simulation of bending the rod assumes that pure bending occurs. There are certain assumptions associated with pure bending. First, it is assumed that the physical shape of the rod maintains an axis of symmetry. This axis is the physical axis of the rod. Secondly, it is assumed that plane sections of the rod, perpendicular to the axis, remain straight and perpendicular to the axis during the bending. Figure 2.3a shows the plane sections before and after bending. The axes of symmetry are x and x' . Note in Figure 2.3b that, as the rod is bent, the segments of the rod experience compression on the concave side of x' , and tension on the convex side. It is assumed that the x' axis is a neutral surface, i.e., there is no stress along this curve. The stress in the upper and lower positions of the rod is proportional to the distance from the neutral surface. The compression and tension will have an effect on the index of refraction which produces an additional index gradient within the rod. The magnitude of the bend-induced index gradient is dependent upon the base glass and the gradient of the rod, and the strength of the bend. The order of magnitude of this effect is calculated in Appendix B for some silicate glass rods and is found to be a significant factor in some cases. This effect is not included in the program developed for the analysis of bent

Rotation of Normal Planes During Pure Bending

Figure 2.3

rods which must be considered when applying the results of the program.

Under the assumption of pure bending, the transformation from the straight rod to the curved can be visualized by a series of rotations and translations of the planes or thin segments of the rod. Figure 2.4 depicts the transformation of a segment of a straight rod to a rod curved in the meridional plane only. The segment is translated by t_x and t_y , and rotated by the angle θ . For a rod curved in the skew plane, a translation, t_z , and a rotation ϕ about the Z axis are also performed.

The segments of the curved rod and the index profiles within the segments are assumed to lie in planes normal to the curved axis in accordance with the assumptions of pure bending. The appropriate plane and the corresponding rotation angles and translation values for a segment are calculated so the arc length along the physical axis of the curved rod from $x = 0$ to $x = x_0$, equals $(x_0 + t_x)$. The translations t_y and t_z equal $Y(x_0)$ and $Z(x_0)$ respectively from Equation (2.13). The rotation angles and the equation of the normal plane are also determined from Equation (2.13).

At any point on the curve, $(Y(x_0), Z(x_0), x_0)$ the equation of the plane normal to the curve can be determined.

Translation and Rotation of a Segment of an Equivalent Straight Rod to the Curved

Figure 2.4

The slopes of the curved axis in the Yx and Zx planes and the rotation angles θ and ϕ can be determined from the parametric equations

$$\begin{aligned}\text{Slope}_Y = \tan\theta = dY/dx &= a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4, \\ \text{Slope}_Z = \tan\phi = dZ/dx &= b_1 + 2b_2x + 3b_3x^2 + 4b_4x^3 + 5b_5x^4.\end{aligned}\tag{2.14}$$

The direction cosines of the curve at the point are

$$\begin{aligned}L = \cos\alpha &= [1 + \tan^2\theta + \tan^2\phi]^{-1/2} \\ M = \cos\beta &= L \tan\theta \\ N = \cos\gamma &= L \tan\phi.\end{aligned}\tag{2.15}$$

The equation of the plane normal to the curve at a point $(Y(x_0), Z(x_0), x_0)$ is given by

$$Lx + MY + NZ + D = 0\tag{2.16}$$

where

$$D = -(Lx_0 + MY_0 + NZ_0),$$

and L, M, and N are the direction cosines of the curve at (Y_0, Z_0, x_0) .

The representation of the curved rod can be visualized as a number of segments (or planes) which are normal to and centered about the curve given by Equation (2.13). Although this representation of the movement of segments is a good visualization of the final construction of the curved rod, it does not precisely represent the necessary manipulation of the index profile. Details of the index transformation require that the index expansion first be discussed.

2.4 THE INDEX REPRESENTATION

The index representation selected for the straight rod is a representation in use at the University of Rochester [2.3]. The index distribution is described as a power series expanded about the origin:

$$n(x, \zeta) = \sum_{n=0} \sum_{m=0} N_{nm} x^m \zeta^n \quad (2.17)$$

where

$$\begin{aligned} \zeta = & aY^2 + bZ^2 + cX^2 + 2dYZ + 2eYX + 2fZX \\ & + gY + hZ + iX + j \quad . \end{aligned} \quad (2.18)$$

The three simplifications of this expansion which are referred to the most frequently are the axial, radial, and anamorphic gradient index profiles.

An axial gradient index medium is one in which the index is a function of x only. Figure 2.5 shows an axial gradient index lens and its planes of constant index. The index equation reduces to

$$n(x) = N_{00} + N_{01}x + N_{02}x^2 + \dots , \quad (2.19)$$

where N_{00} is the base index of the glass and N_{01} , N_{02} , $\dots N_{0m}$ are the coefficients of the profile. An axial gradient does not have focusing power, but it provides useful degrees of freedom for aberration correction in lens design.

The index distribution of a radial gradient is a function of the radial distance from the axis. The index expansion simplifies to

$$n(\zeta) = N_{00} + N_{10}\zeta + N_{20}\zeta^2 + \dots , \quad (2.20)$$

where

$$\zeta = Y^2 + Z^2$$

for $a = b = 1$.

AXIAL GRADIENT-INDEX LENS

$$N = N_{00} + N_{01}x + N_{02}x^2 + \dots$$

Index Expansion and Surfaces of Constant Index
of an Axial Gradient

Figure 2.5

N_{00} is the base index of the glass, and $N_{10}, N_{20}, \dots, N_{n0}$ are the coefficients of the profile. Figure 2.6 shows a radial gradient index Wood lens (a radial gradient lens with plane surfaces), and its surfaces of constant index. The arrows indicate the direction of increasing index. The index decreases with increasing distance from the axis, which results in a positive power Wood lens with power equal to

$$\phi_{\text{gradient}} = -2N_{10}t . \quad (2.21)$$

An imaging GRIN rod is a thick Wood lens, therefore the period length, L , is also determined by the N_{10} term and is given by

$$L = \left[\frac{-2\pi^2 N_{00}}{N_{10}} \right]^{\frac{1}{2}} . \quad (2.22)$$

The higher order terms of the index are used for aberration correction.

An anamorphic Wood lens is depicted in Figure 2.7. The planes of constant index are elliptical cylinders, which results in the focal length in the meridional plane being different from the focal length in the sagittal plane. The index equation reduces to

RADIAL GRADIENT-INDEX LENS

$$N = N_{00} + N_{10} (Y^2 + Z^2) + N_{20} (Y^2 + Z^2)^2 + \dots$$

$$\phi_{\text{grad}} = -2 N_{10} t$$

Index Expansion and Surface of Constant Index
of a Radial Gradient
Figure 2.6

ANAMORPHIC GRADIENT INDEX LENS

$$N = N_{00} + N_{10}\zeta + N_{20}\zeta^2 + \dots$$

Index Expansion and Surfaces of Constant Index
of an Anamorphic Gradient
Figure 2.7

$$n(\zeta) = N_{00} + N_{10}\zeta + N_{20}\zeta^2 + \dots, \quad (2.23)$$

where

$$\zeta = aY^2 + bZ^2$$

$$a \neq b$$

and $N_{00}, N_{10} \dots N_{n0}$ are as previously defined. A lens with an anamorphic profile is useful as a one dimensional lens. An anamorphic gradient index distribution is asymmetric and, therefore, does not have an axis of rotational symmetry.

Including all of the terms of ζ in the index representation allows the surfaces of constant index to be written as any quadratic surface in space. It is instructive to consider the terms of ζ in three groups. The first three terms, $aY^2 + bZ^2 + cX^2$, can describe any quadratic surface centered about the origin. The cross terms, $2dYZ + 2eYX + 2fXZ$, allow the quadratic to be rotated. The last four terms, $gY + hZ + iX + j$, are used in the translation of the surface.

The terms of ζ are used to describe the asymmetric or symmetric index distribution of the straight rod and then are modified to generate the translations and rotations needed to construct the index profile of a segment of the curved rod. The modified expansion represents the index profile of a segment of the curved rod and not the entire

rod. An analytic expression of the index for the entire rod is not necessary for the ray trace method selected. Only the value of the index and its gradient are needed at each incremental step.

Figure 2.8 depicts the transition of the index profile of an equivalent straight rod to generate the appropriate index profile in a plane of the curved rod. The value of the index at any point (Y, Z, x) in the curved rod is calculated by first determining the normal plane in which the point is contained. The index profile of this plane is generated by modifying ζ in the index equation of the equivalent straight rod so that the equivalent planes (or segments) overlap. As defined previously, the equivalent planes are the planes containing the physical axis points of equal arc lengths. In Figure 2.8, the arc length from $x = 0$ to $x = x_0$ along the curved axis equals x_1 . Moving the equivalent plane of the straight rods requires the rotation and translation of the index profile; i.e., the modification of the ζ terms. Since the index is expanded about the origin, indicated by $[0]$ in Figure 2.8, this is the point about which the rotations and translations occur. The index distribution resulting from the modification of ζ is shown in the bottom portion of Figure 2.8. The visualization of the curved rod as a series of segments normal to and centered about the curved axis is

The Rotation and Translation of the Index Expansion
for Equivalent Planes
Figure 2.8

correct but incomplete. Since the index distribution is an expansion about the origin, the visualization of the index profile of the constructed curved rod is a series of straight rods with gradually increasing rotations and displacements from the origin.

2.5 MODIFICATION OF THE ζ TERM

Techniques from linear algebra are implemented to modify the terms of ζ in order to produce the rotations and translations of the index distribution as described in the previous sections. ζ can be written in matrix notation as:

$$\begin{aligned} \zeta = & aY^2 + bZ^2 + cX^2 + 2dYZ + 2eYX + 2fZX \\ & + gY + hZ + iX + j \end{aligned} \quad (2.24)$$

$$= [Y \ Z \ X] \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \begin{bmatrix} Y \\ Z \\ X \end{bmatrix} + [g \ h \ i] \begin{bmatrix} Y \\ Z \\ X \end{bmatrix} + j \quad (2.25)$$

$$= X^t Q X + L X + j$$

where x is a position vector from the origin, x^t is the transpose of x , Q is the matrix of second order terms of ζ , L is the vector of linear terms of ζ , and j is a constant.

To translate the index profile (i.e., the origin) by a value t_y , t_z or t_x , replace Y , Z , or x by $(Y - t_y)$, $(Z - t_z)$ and $(x - t_x)$ respectively. This substitution has no affect on the coefficients of the second order terms of ζ and produces the following modifications of the linear and constant terms:

$$\begin{aligned}
 g' &= g - 2at_y - 2dt_z - tet_x \\
 h' &= h - 2dt_y - 2bt_z - 2ft_x \\
 i' &= i - 2et_y - 2ft_z - 2ct_x \\
 j' &= j + at_y^2 + bt_z^2 + ct_x^2 + 2dt_y t_z + 2et_y t_x \\
 &\quad + 2ft_x t_y - gt_y - gt_z - it_x.
 \end{aligned}
 \tag{2.26}$$

Modifications to ζ for the rotation of the index profile is accomplished by the use of rotation matrices, matrices which perform rotations from one orthogonal coordinate

system to another. These matrices are denoted by P_Y , P_Z , and P_x where the subscript indicates the axis about which the rotation occurs. The transpose of the matrices are denoted by P_Y^t , P_Z^t , and P_x^t . The rotation matrices can be found in Table 2.1. Since the assumption of pure bending has been made only rotations about Y and Z are considered. The index distribution of a straight rod is given by Equation (2.25) in the (Y, Z, x) coordinate system. The index distribution is rotated by rotating the coordinate system and the position vector x to an x' . Using the appropriate P matrix, the modification to x due to the transition between coordinate systems is given by

$$x = Px' \quad (2.27)$$

and

$$x' = P^{-1} x . \quad (2.28)$$

Since the rotation matrices are orthogonal matrices, it follows that

$$P^{-1} = P^t$$

and Equation (2.28) becomes

$$x' = P^t x . \quad (2.29)$$

$$P_Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \quad P_Y^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$P_Z = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \quad P_Z^t = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

$$P_X = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_X^t = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The Rotation Matrices

Table 2.1

Substituting Equation (2.27) into (2.25) gives the equation of the index distribution in the rotated coordinate system

$$(Px')^t Q (Px') + L (Px') + j . \quad (2.30)$$

Using the property of the transpose operation which states $(AB)^t = B^t A^t$, Equation (2.30) becomes

$$(x'^t P^t) Q (Px') + L (Px') + j \quad (2.31)$$

The associative property of matrix multiplication allows the order of multiplication to be switched to a more convenient form which is

$$x'^t (P^t Q P) x' + (LP) x' + j \quad (2.32)$$

or

$$x'^t Q' x' + L' x' + j$$

where

$$Q' = P^t Q P$$

$$L' = LP .$$

The index distribution in a plane of the curved rod is generated by first calculating the Q' and L' matrices to perform the rotation by θ and ϕ . The rotated rod is then

translated using Equation (2.26). The modified ζ term results in an index expansion which describes the index in the plane of the curved rod under consideration. The index at a point anywhere in this plane can be calculated by substituting the (Y, Z, x) coordinates of the point into the modified index equation. Since the index can be calculated at any point within the curved rod, the gradient of the index, ∇n , can be determined numerically at any point. The necessary information to implement the ray trace is available.

2.6 DATA INPUT TO PROGRAM

The program requires a lens deck containing data for the equivalent straight rod and the curve under consideration (see Appendix A). The information from the design of the equivalent straight rod system must include the surface by surface data of the object position and height, the rod's length, clear aperture, base index and gradient index distribution, and the location and size of the entrance pupil. The coefficients of Equation (2.13) for the bending and the extrapolation distance Δt to be used by the ray trace algorithm are also user specified. The completed lens deck is read by the program and the curved GRIN rod system is then available for the analysis options indicated in the evaluation section.

2.7 EVALUATION OF THE BENT GRIN ROD

Standard techniques of image analysis for optical systems require the use of paraxial ray tracing to determine the perfect image. Paraxial rays are rays which use the first order approximation of the ray path, and therefore do not exhibit any aberrations. The real ray is compared to the perfect, or paraxial, ray as a measure of the imaging ability of the system. Since the ray tracing program developed can only trace real rays, determining the paraxial image plane is not straightforward. Paraxial rays are simulated in the program by tracing rays from small object heights and at small angles. Checks are performed by the program to verify the acceptability of the simulation.

In order to determine the paraxial image plane for a curved rod system, some alternate definitions for the paraxial optics of a system must be established. The optical axis of the bent rod is not easily defined, therefore in its place an axial ray is used. An axial ray is defined as the ray launched from the axial object point and parallel to the x axis. (see Figure 2.9) A paraxial ray is a ray launched from an object point slightly off axis at a small inclination to the axial ray.

The image plane is determined by two rays, the axial ray and a meridional paraxial ray with a positive slope from the

The Definition of the Paraxial Image Plane

axial object point (see Figure 2.9). The point of intersection in image space of these two rays is the perfect axial image point. In some cases, such as the end-to-end imaging depicted in Figure 2.9, this is a virtual image. The paraxial image plane is defined as the plane which contains this intersection point and is normal to the axial ray in image space. For the case of a straight rod, with end to end imaging, the axial ray, the physical axis of the rod, and the optical axis coincide with the x axis. The paraxial image plane is therefore on the back endface of the rod. As the rod is bent the axial ray deviates from the physical axis and the image plane shifts and tilts with respect to the end face. The definition of the axial ray and image plane account for the shift and tilt, thereby determining the effects of bending on the first order properties of the rod separately from the aberrations in the image plane. A number of checks are provided by the program to test the accuracy of this definition of the paraxial image plane. In most of the cases studied it is acceptable. The exception is noted in Chapter 3.

It is useful to establish a local coordinate system in the image plane when dealing with ray deviations from a perfect image point. The coordinate system lies in the image plane and its origin is located at the axial image

point. The axis are labeled ϵ_y and ϵ_z as shown in Figure 2.9.

The paraxial magnification is assumed to be equal to the meridional plane magnification and is given as the ratio

$$M = h'/h, \quad (2.33)$$

where h is the height of a meridional object point slightly off axis, and h' is the height of the image of this point in the image plane (see Figure 2.10). The image of the off axis point is determined by the ray from the object point launched parallel to the x axis. The value of h' is the distance from this image point to the position of the axial ray in the image plane.

The perfect image of any object point (Y_o, Z_o) is the point $\epsilon_y = MY_o, \epsilon_z = MZ_o$. The absolute coordinates in the (Y, Z, x) system can also be determined, but are not as informative as the local coordinates when analyzing the aberrations.

The technique used to quantify the aberrations of the curved rod system is the plotting of the transverse aberrations for a specified field angle, i.e. a specified object height, as a function of the entrance pupil coordinate. This plot is referred to as the transverse aberration plot from a specified fractional object height

The Definition of the Paraxial Magnification

Figure 2.10

(f.o.b.) , or fraction of the maximum field angle. The transverse aberration is the difference in the image plane between the real ray and the real chief ray. A series of rays are launched from a specified object point toward evenly spaced points in the entrance pupil, as shown in Figure 2.11 and 2.12. The location and size of the entrance pupil of the equivalent straight rod are required input to the program. It is assumed that the entrance pupil does not shift or tilt with bending. Since the entrance pupil is only used as a means of specifying the launch angles of the rays from the object, and the object is always normal to the x axis, there is no loss of generality to the program in this assumption. The series of rays most frequently used are referred to as the meridional, or Y, fan and the sagittal, or Z, fan. Fans at other inclinations in the entrance pupil are used to determine certain types of aberrations which are not present in the meridional or sagittal fans.

The chief ray is the ray which passes through the center of the entrance pupil. Referencing the transverse aberrations to the chief ray rather than the perfect image point results in subtracting the real distortion from the aberration plots. At a given object height the real distortion is the deviation of the chief ray from the

The Rays Traced in a Meridional and Sagittal Fan

Figure 2.11

45° FAN

135° FAN

The Rays Traced in

45° and 135° Fans

Figure 2.12

perfect image point. The only effect real distortion has on the aberration plots is the addition of a constant value. The value of the real distortion is given separately from the transverse aberration data.

The deviations of each ray in the fan, from the chief ray is plotted against the normalized radius in the entrance pupil, $\bar{\rho}$, (see Figure 2.11). The plots from various object heights are used to determine the type and magnitude of the aberrations present. The transverse aberration plots, as specified, will indicate all the aberrations except distortion. The changes in first order properties will not affect the plots due to the specification of the image plane. The factors of image plane tilt and shift, and distortion are calculated and presented separately.

Another option available for the image analysis of the GRIN rod system is a spot diagram. Twenty five to one hundred rays from a specified object point aimed at evenly spaced coordinates in the entrance pupil are traced. A point plot of the deviation in the image plane of these rays from the chief ray is the spot diagram. The spot diagram is a good visual aid of the optical system's ability to image a point.

As previously mentioned, the curved imaging GRIN rod is not a rotationally symmetric optical system. Therefore,

even and odd order aberrations are present. In the analysis of the total aberration, it is instructive to consider the contributions of the lower orders. In many cases, the lower order aberrations dominate and the coefficients of the first, second, and third order transverse aberrations can be calculated from the aberration plots.

The transverse aberration equations for a non-rotationally symmetric optical system were derived by Sands [2.4]. In the most general form the equations contain many terms which make conceptual interpretations of the aberrations difficult. To reduce the number of terms for the analysis of the lower order aberrations, two restrictions are placed on the optical system. First, the object is assumed to always lie in the meridional plane. Secondly, the system is assumed to exhibit plane symmetry. This assumption requires that the object and the bend of the rod lie in the same plane; i.e., the meridional plane.

The first, second, and third order transverse aberration equations for a system with the above restrictions are given in Table 2.2. The first order aberrations contain terms of defocus, linear astigmatism, and linear distortion. Using Sands notation, an astigmatic aberration is one which is a function of odd powers of $\bar{\rho}$, the normalized radius in the entrance pupil. Distortion terms are functions of powers of

$$\epsilon_{1Y} = \psi_1 \bar{\rho} \cos \theta + \ell \bar{\rho} \cos \theta + \psi_3 \bar{h}$$

$$\epsilon_{1Z} = \psi_2 \bar{\rho} \sin \theta + \ell \bar{\rho} \sin \theta$$

where: ψ_1, ψ_2 - first order linear astigmatism

ℓ - defocus

ψ_3 - linear distortion

$$\epsilon_{2Y} = (\omega_1 + \omega_2 \cos 2\theta) \bar{\rho}^2 + \omega_4 \bar{\rho} \bar{h} \cos \theta + \omega_6 \bar{h}^2$$

$$\epsilon_{2Z} = \omega_3 \bar{\rho}^2 \sin 2\theta + \omega_5 \bar{\rho} \bar{h} \sin \theta$$

where: $\omega_1, \omega_2, \omega_3$ - second order, zeroth degree coma

ω_4, ω_5 - second order, linear astigmatism

ω_6 - second order distortion

$$\epsilon_{3Y} = \tau_1 \bar{\rho}^3 \cos \theta + \tau_2 \bar{\rho}^3 \cos \theta \sin \theta + (\tau_5 + \tau_6 \cos 2\theta) \bar{\rho}^2 \bar{h} + \tau_8 \bar{\rho} \bar{h}^2 \cos \theta + \tau_{10} \bar{h}^3$$

$$\epsilon_{3Z} = \tau_3 \bar{\rho}^3 \sin \theta + \tau_4 \bar{\rho}^3 \cos^2 \theta \sin \theta + \tau_7 \bar{\rho}^2 \bar{h} \sin 2\theta + \tau_9 \bar{\rho} \bar{h}^2 \sin \theta$$

where: $\tau_1, \tau_2, \tau_3, \tau_4$ - Third order, third degree astigmatism

τ_5, τ_6, τ_7 - Third order, linear coma

τ_8, τ_9 - Third order linear astigmatism

τ_{10} - Third order distortion

The Transverse Aberration
Polynomial Up to Third Order.

Table 2.2

$\bar{h}$ only, the normalized object height. The defocus term accounts for a shift of the image plane and is not present in most systems. The second order terms include aberrations of the astigmatic type, the ρh term, and distortion, the $\bar{h}^2$ term, as well as the comatic type. A comatic aberration is a function of even powers of $\bar{\rho}$.

For a focused, rotationally symmetric system, the first and second order terms are zero, and the third order terms reduce to the Seidel aberrations. The analogous coefficients of the asymmetric system and the Seidel coefficients are as follows:

$\tau_1 \rightarrow \sigma_1$	spherical
$\tau_5, \tau_6, \tau_7 \rightarrow \sigma_2$	coma
$\tau_8, \tau_9 \rightarrow \sigma_3, \sigma_4$	field curvature
$\tau_{10} \rightarrow \sigma_5$	distortion .

The evaluation of the aberration plots from a few object heights provide the data necessary to determine the transverse aberration coefficients. The coefficients, up to third order, have been calculated for some GRIN imaging rods with a variety of bends. Some examples and the noted trends are presented in Chapter 3.

Chapter III

EXAMPLES OF BENT GRIN RODS

3.1 THE EQUIVALENT STRAIGHT ROD

An equivalent, straight, radial gradient index rod with end-to-end imaging was selected as the rod for the bending study. The rod, which will be referred to as GD3, is designed as a long image relay with diffraction limited performance over a length of 1.0m [3.1]. The physical parameters and the first order properties of GD3 are given in Table 3.1.

The design of the rod was optimized by modifying the index profile, $f/\#$, and rod diameter in order to minimize the transverse aberrations. Since the straight rod is a rotationally symmetric system, only odd order aberrations are present. The total aberration over a number of half periods is equal to the aberration at a half period, times the number of half periods traversed. The number of half periods in the 1.0m rod is 27. The design optimization was therefore performed on a half period segment in the limit that the transverse aberrations, at the half period image,

PHYSICAL PARAMETERS

$$N_{00} = 2.4$$

$$N_{10} = -.00865 \text{ mm}^{-2}$$

$$N_{20} = .000034 \text{ mm}^{-4}$$

$$\Delta n = .0027$$

$$\text{Radius of Rod} = .55 \text{ mm}$$

$$\frac{1}{2} \text{ Period Length} = 37 \text{ mm}$$

FIRST ORDER PROPERTIES

Object/Image on Endfaces of Rod

Maximum field angle = .08333 radians

$$f/\# = 6$$

Maximum object height = .375 mm

Physical Parameters and First Order
Properties of the GD3 Rod
Table 3.1

had to be less than 1/27th of the diffraction spot size. The diffraction spot diameter of a GRIN rod is

$$d = 2.44(f/\#)\lambda . \quad (3.1)$$

The spot diameter for the one meter length rod is $14.6\mu\text{m}$, and the limit for the half period spot radius is $0.27\mu\text{m}$.

The transverse aberrations plots from the fractional field angles of 0.0, 0.6, and 0.9 for the half period image are presented in Figure 3.1. The ϵ_y values correspond to data from meridional fans and the ϵ_z from sagittal fans. A least squares fitting routine is used to fit the data from the plots to the transverse aberration equation up to third order, (see Table 2.2). Since the system is rotationally symmetric the second order aberrations are zero. The calculated third order Seidel coefficients are

$$\sigma_1 = 0.141\mu\text{m}$$

$$\sigma_2 = 0$$

$$\sigma_3 = 0.257\mu\text{m}$$

$$\sigma_4 = -0.652\mu\text{m}$$

$$\sigma_5 = 0 .$$

The third order coma and distortion are zero because of the symmetry of the system. The aberrations present are spherical aberration and field curvature. Since normalized

Transverse Aberration Plots of the Straight
Half Period GD3 Rod
Figure 3.1

coordinates, $\bar{p}$ and $\bar{h}$, are used in the fit to the aberration polynomial, the coefficients carry the units of length of the system.

3.2 EFFECTS ON THE FIRST ORDER PROPERTIES DUE TO BENDING

The paraxial image plane and the paraxial magnification, as defined in section 2.7, are examined to determine any changes resulting from the bending of the rod. The equivalent straight rod is a half period length of GD3. The object on the front face of the rod images to the back endface and is centered about the physical axis (see Figure 3.2a). As the rod is bent, for example, along a cubic curve, the ray paths shift and the image is displaced off of the physical axis (see Figure 3.2b). The ray bundles in Figure 3.2 depict the ray path and image of the axial object point for the two cases. Since the magnification at the half period is -1, the image is inverted, and for the bent rod the upper portion of the image is vignetted.

The aberration plots for the straight, half period rod were determined in the image plane on the back endface. If it is assumed that the first order properties are not affected by the bending, the image plane of the curved rod would remain on the back endface and the magnification would be -1.0. The aberration plots for three meridional fans

Imaging of the Axial Object Point in a
Straight and Curved Rod

Figure 3.2

from fractional field angles of 0.0, 0.6, and 0.9 are determined in the plane of the endface, and are shown in Figure 3.3.

The displacement of the axial image point from the physical axis and the distortion have been subtracted from the plots. These factors, if not subtracted, would appear as a constant value added to the plots. The transverse aberrations for the straight rod are given by the solid line. The aberrations for two cubic bends are given in the dashed lines. The stronger of the two cubic curves corresponds to the rod shown in Figure 3.2b. The rotation of the axial fan indicates that the real image plane focal shifts into the rod, and that the amount of focal shift increases with greater bending. The vignetting of the image is detected in the off axis plots from fractional field angles equal to 0.6 and 0.9. For the milder bend, the 0.6 object point is imaged, but the 0.9 point is partially lost. For the stronger bend, the 0.6 and 0.9 points have been totally vignetted.

The shift of the image plane noted above and any change in the magnification of the bent rod system is accounted for by use of the alternate definitions of the paraxial image plane and magnification, as presented in Section 2.7. The movement of the image plane will be described as a focal

$$Y = a_3 x^3$$

$a_3 = .00$	—
$= .0000006$	- - -
$= .0000013$	- - -

Transverse Aberration Plots for GRIN Rod
Bent Along a Series of Cubic Curves
Figure 3.3

shift and a displacement in the local coordinate system, (Y', Z', x') , of the back endface of the rod. Figure 3.4 shows the Yx plane of the endface of a bent rod. The image of the axial object point is located at P. The plane normal to the axial ray, and containing the point P, is the image plane, as indicated by the short dashed lines. The shift of the point P from the physical axis, labeled O, is described by x_s' and Y_s' . The Z_s' component is not shown in the Figure. The tilt of the image plane, θ_s , is calculated relative to the endface and as shown is considered a positive rotation.

To study the shifts and rotation of the paraxial image plane as a function of the bending, the 37mm half period, GD3 rod is bent along a curve in the meridional plane, so that the physical axis at the endface is displaced from the x axis by a specific value. The described displacement is indicated by CAY in Figure 3.2b. The endface is allowed to reach a series of specified displacements by bending the rod along a series of parabolic and cubic curves. The values of Y_s' and x_s' are plotted against the CAY values for the two types of curves (see Figure 3.5). Both the parabolic bends, indicated by the solid line, and the cubic bends, by the dashed lines, exhibit the same trend. The stronger the bend is, the larger the image displacement and

The Displacement and Rotation of the Image Plane

Figure 3.4

The Variation of Focal Shift and Displacement With
Stronger Bending of the Rod

Figure 3.5

focal shift. In each case the cubic curve results in a larger shift than the parabolic. The plots for the cubic bend stop at CAY approximately equal to 0.8mm, since beyond this bend the axial ray is vignetted. The parabolic curve allows a greater CAY displacement before the axial ray is vignetted.

The tilt of the paraxial image plane relative to the endface, θ_s , is negligible for all the parabolic bends examined. The value of θ_s for the cubic bends varies linearly with the value of CAY, and is approximately 3° when the axial ray is vignetted.

The magnification remains very close to 1.0 in all cases of the bending. A slight change is noticed for larger focal shifts of the image into the rod. A real image is formed within the rod and the light diverges outside the rod as a result to the by the small remaining length of the rod. This causes a virtual image inside the rod. The imaging by the small end section of the rod causes a slight change in the magnification but the change, for the cases studied here, is negligible.

The elimination of the effects of shifts and rotations of the image plane from the aberration plots facilitates the evaluation of the transverse aberrations. In the following sections, the transverse aberrations and the aberration

coefficients of the rod bent along the stronger of the parabolic curves are examined.

3.3 BENT ROD: PARABOLIC

To examine the transverse aberrations of a bent rod, the GD3 relay rod discussed in the previous section is again considered. A comparison of the aberrations of the straight rod and the aberrations of a rod bent along a simple curve, a parabola, is used to determine the effects of the asymmetry of the system. The parabolic curve lies in the meridional plane and is given by

$$Y = 0.00048x^2 . \quad (3.2)$$

The coefficient of the curve was chosen to be large enough to introduce significant bend-induced aberrations, but not to cause significant vignetting and prevent the analysis. Limiting the curve to the meridional plane establishes the plane of symmetry necessary to allow the fitting to the lower order aberration polynomial (see Table 2.2).

Rods with lengths equal to $1/2$, 1, $1\frac{1}{2}$, and 2 periods are analyzed. Recall that the half period length is 37mm. Figure 3.6 shows the ray path of the axial ray and the bundle of rays from the axial object point for a rod with length equal to 1 period.

Imaging the Axial Object Point
in a Parabolic Rod

Figure 3.6

At the half period, the image is displaced off the physical axis. Since the magnification at the half period is -1, the upper portion of the image is totally vignetted. The Y's displacement is -0.266mm and the focal shift, x's, is -4.2 μ m (a focal shift into the rod). The rotation of the image plane relative to the endface of the rod is negligible, being less than 0.01°.

At the full period, the image is approximately on the physical axis, and the magnification is +1. The Y's displacement is 0.001mm, and the focal shift, x's, is -27.9 μ m. The rotation of the image plane has increased, but is still negligible.

An interesting feature to note in Figure 3.6 is that the axial object point images to the displaced point at the half period, and then images back onto the physical axis at a full period. An understanding of this imaging phenomena is achieved by considering a rod of half period length. Since the maximum object height is 0.375mm, the displacement of -0.266mm corresponds to a fractional object height of -0.709. In any optical system, the roles of the object and image can be reversed, therefore it holds that a ray traced in an optical media must follow the same path whether it is traced forward or backward. Applying this principle to the half period image of the rod, it must hold true that if a

point from a fractional object height of 0.0 images to a fractional object height of -0.709, a point from a fractional object height of -0.709 must image to a fractional height of 0.0. Since the parabolic curve of the rod in the region of the first one or two periods closely approximates the arc of a circle, each half period segment images in an identical manner. The first half period images the axial object point to the -.709 fractional height, and the second half period, since it is identical, must image this off axis point to a fractional height of 0.0. This results in the axial object point imaging on to and off of the physical axis in this region. The same imaging effect is also seen in the second period of the rod where the Y_s' displacement at the 1 1/2 and 2 period length rods are -0.266mm and 0.004mm respectively.

This imaging characteristic is predicted by Rohra and Thyagarajan [3.2] in the derivation of the second and third order aberrations coefficients of a radial GRIN imaging rod curved along the arc of a circle. The first section of the parabola closely approximates the arc of a circle, therefore the analytic results of this study can be used as a check on the ray tracing program. Figure 3.7 shows a rod curved along the arc of a circle and two rays which are noted by the study. The first ray is a ray launched from the axial

Two Ray Paths in a Rod Curved about
the Arc of a Circle

Figure 3.7

object point parallel to the x axis, i.e. the axial ray. The path this ray follows is described by

$$\Delta Y = (1 - \cos(\alpha s))/R\alpha^2 , \quad (3.3)$$

where

$$\alpha^2 = -2N_{10}/N_{00} ,$$

ΔY is the distance from the physical axis to the ray path, s is the arc length along the physical axis, and R is the radius of curvature of the circle. Images of the axial object point are located at

$$s = m(\pi/\alpha), \quad m = 1, 2, \dots . \quad (3.4)$$

The second ray is the ray launched parallel to the x axis at the height of

$$Y = 1/(R\alpha^2) . \quad (3.5)$$

This ray travels along a path at a constant distance from the physical axis of the rod. The value of Equation (3.5) for the parabolic GD3 rod is $-0.1335\mu\text{m}$. A ray traced by the program at this height remains at this constant distance from the physical axis for the 2 period length rod. The image displacements of the axial ray traced by the program at the half and full period length rods are also in agreement with the values predicted by Equation (3.3).

Another interesting feature of this imaging characteristic is found by considering the transverse aberrations of the 0.0 and -0.709 fractional field angle points. The correspondence in the imaging of these two points implies that the aberrations at the image of these points must be identical. This is verified by the raytracing program.

The transverse aberrations plots of the half period length rod for several object heights, and a plot of the distortion are given in Figure 3.8. The two axial plots, in Figures 3.8a and 3.8b, present the data of the sagittal and meridional fans respectively. For a meridional fan, the ϵ_z value is always zero. For a sagittal fan, establishing the meridional plane as the plane of symmetry causes the ϵ_y and ϵ_z plots to be anti-symmetric. The ϵ_y of the sagittal fan indicates the presence of an aberration which is linear in $\bar{p}$, and independent of $\bar{h}$. This same aberration is therefore present in all the sagittal fans, but has only been plotted in the axial one. The plots for the off-axis object points have two curves. The solid curve indicates the ϵ_y values of the meridional fan, and the dashed show the ϵ_z values of the sagittal fan. In some cases the aberration curves have been truncated with a slash to indicate that the rays with larger values of $\bar{p}$ (or $\bar{h}$) were vignetted. The fan from a

The Transverse Aberration Plots at the Half Period Image Plane

Figure 3.8

fractional field angles equal to 0.9 is totally vignetted. The distortion plot, Figure 3.8f, gives the value of the distortion for the various object heights.

Some general conclusions can be drawn from the transverse aberration and distortion plots. The asymmetry of the meridional and distortion plots indicates the presence of even order aberrations. The rotation of the axial sagittal fan demonstrates a focal shift that is not present in the meridional plane. This is referred to as first order, linear astigmatism. This focal shift is always evident in the sagittal fans since the image plane is always placed at the meridional focus. Further analysis of the transverse aberrations is facilitated if the data is fitted to the aberration polynomial, up to the third order, given in Table 2.2. To perform the fit it is necessary to trace some rays with θ equal to a value other than 0° , or 90° . This is completed and the results of the fit are found in the second column of Table 3.2.

The third order coefficients in Table 3.2 are given in the general form. It should be noted that these coefficients are the same as the Seidel coefficients of the equivalent straight rod, presented in section 3.1. The values of τ_1 and τ_3 are equal to σ_1 , τ_8 to $(3\sigma_3 + \sigma_4)$ and τ_9 to $(\sigma_3 + \sigma_4)$. The third order aberrations of this system do not change significantly from the bending.

<u>Order</u>	<u>Coefficient*</u>	<u>$\frac{1}{2}$ Period</u>	<u>1 Period</u>	<u>$1\frac{1}{2}$ Periods</u>	<u>2 Periods</u>
1st	ψ_1	0	0	--	--
	ψ_2	$-.077\mu\text{m}$	$.156\mu\text{m}$	$-.230\mu\text{m}$	$.325\mu\text{m}$
	ψ_3	.018	$-.006$	.027	$-.028$
2nd	ω_1	$-.032$	0	$-.032$	0
	ω_2	$-.016$	0	$-.016$	0
	ω_3	0	0	--	--
	ω_4	.082	$-.167$	--	--
	ω_5	$-.282$	.567	--	--
	ω_6	$-.156$	0	$-.154$	0
3rd	τ_1	.143	$-.290$	.448	$-.620$
	τ_2	0	0	0	0
	τ_3	.143	$-.289$	.446	$-.617$
	τ_4	0	0	0	0
	τ_5	0	0	--	--
	τ_6	0	0	--	--
	τ_7	0	0	--	--
	τ_8	.119	$-.237$	--	--
	τ_9	$-.395$	.790	--	--
	τ_{10}	0	0	0	0

*See Table 2.2 for a definition of the coefficients.

The Aberration Polynomial Coefficients for a GRIN Rod
Curved Along a Parabolic Curve

The changes in the transverse aberrations are due to the addition of the first and second order terms. The first of these terms, ψ_1 , will always be zero since the meridional focus is selected as the image plane by definition. The value of ψ_2 equals $-0.077\mu\text{m}$, and is the first order, linear astigmatism. The second order terms, ω_1 - ω_6 , which vary as $\bar{\rho}^2$, $\bar{\rho}\bar{h}$, and $\bar{h}^2$, are presented in the table.

The linear distortion term, ψ_3 , which equals $0.021\mu\text{m}$ is caused by an error in calculating the paraxial magnification. The simulation of a paraxial ray by tracing a real ray at a small field angle results in an error in the magnification calculation. One effect of the error in magnification is to introduce a linear in $\bar{h}$ term into the aberration polynomial. The aberrations which are functions of $\bar{\rho}$ alone are not affected by the error. The aberrations which are functions of $\bar{\rho}\bar{h}$ are altered, but the effect is insignificant if the ψ_3 coefficient is of the same order of magnitude as the other aberrations. The value of ψ_3 for the rods in Table 3.2 is acceptable and indicates that a close approximation of the paraxial magnification was calculated and used in the evaluation.

The rod with a length equal to one period was examined in a manner similar to the half period rod. The transverse aberration plots are in Figures 3.9. The ordinate axis

The Transverse Aberration Plots at the One
Period Image Plane
Figure 3.9

scale of these plots is twice the scale of the plots previously given. The magnification at this image plane is +1.0, therefore, the aberrations have the opposite sign than those of the half period. The meridional fans do not have the asymmetries of the half period plots, and the first order, linear astigmatism is zero. The distortion plot is not presented since the amount is insignificant.

The axial image point is located on the physical axis of the rod at the full period, and all the rays in the fans have traveled on both sides of the axial ray, i.e. on the $+\bar{p}$ and $-\bar{p}$ sides. This introduces a certain amount of symmetry in the imaging, which can be noted in the vignetting of the system. In the half period rod, all the vignetted rays of the meridional fan have $-\bar{p}$ coordinates. During the second half period the same vignetting happens to the opposite ray in the fan, producing a balanced vignetting of the $+\bar{p}$ and $-\bar{p}$ rays.

The coefficients from the fit to the aberration polynomial are given in the third column of Table 3.2. The third order coefficients have all doubled in magnitude and changed sign as is expected in a straight rod. The ψ_2 , ω_4 , and ω_5 terms have approximately followed the same trend. The ω_1 , ω_2 , and ω_6 terms are zero.

The coefficients of the distortion and axial aberrations were determined for a 1 1/2 and 2 period length rod. (See Table 3.2) The imaging of the axial object point on and off the axis introduces a symmetry which causes the $\bar{p}^2$ and $\bar{h}^2$ terms to go to zero at each full period. The other terms in the polynomial approximately double with each half period. This doubling and canceling of terms is a good approximation for a limited number of periods. In the second period image plane the small differences between τ_1 and τ_3 indicate either that higher order aberrations may be summing over the half periods to become noticable, or that the longer length rod is no longer in the region of limited symmetry.

3.4 BENT ROD: ANTI-SYMMETRIC

Further insight into the trends of the even order aberrations from various bendings of a GRIN rod is gained by considering a rod which has some opposite symmetries to the parabolic rod previously discussed. The parabolic GD3 rod consisted of a series of positively sloped half periods. A rod which approaches the opposite of this symmetry is one which has an inflection point at the half period, and is anti-symmetric about the inflection point.

The curve selected for such a rod is described by

$$Y = 0.219x^2 - 0.987x^3 + 0.16x^4 - 0.865x^5 . \quad (3.6)$$

Once again GD3 is the equivalent straight rod. The slope of the curve is zero at $x = 0$, $x = 37\text{mm}$, and $x = 74\text{mm}$, therefore, the front and back endfaces are normal to the x -axis. The displacements of the physical axis from the x -axis at the half and full period are 0.4mm and 0.8mm respectively. Figure 3.10 depicts the curved GD3 rod, which will be referred to as the anti-symmetric rod, and the axial imaging bundle of rays.

The image plane of the half period length rod has a focal shift, x_s' , of $0.78\text{ }\mu\text{m}$ and a Y_s' displacement of 0.131mm . The angle of the image plane relative to the back endface is -3.5° . At the full period length, the values of x_s' and Y_s' are $5.55\text{ }\mu\text{m}$ and $-.262\text{mm}$ respectively, and the value of θ_s is negligible.

The transverse aberration plots for the half and full period image are not presented; instead the evaluation of the aberration trends is made from the fitted coefficients. The only significant point which is evident from the plots and not the coefficients concerns the vignetting. For most object points all the vignetting occurs in the first half period. This is evident from the axial bundle of rays in Figure 3.10. The vignetted rays in the fans are rays launched to the $-\bar{p}$ coordinates. Additional vignetting does not occur in the second half period due to the inflection

Imaging the Axial Object Point
for the Anti-Symmetric Rod

Figure 3.10

point; therefore, the vignetting is not balanced as it was in the case of the parabolic bend. For object points at the edge of the field, a small number of rays launched to $+p$ coordinates are lost, but the vignetting is still not balanced at the full period image.

The fitted aberration polynomial coefficients for the half and full period image plane are presented in Table 3.3. The trends of the anti-symmetric rod are not as easily definable as in the parabolic rod example. The third order aberrations are still the Seidel aberrations and have again doubled in magnitude and changed sign. The third order coma, τ_5 , τ_6 , and τ_7 , and distortion, τ_{10} , are still zero. The first order term, ψ_1 , is zero as it should be, and the order of magnitude of ψ_3 indicates that the paraxial region was used in determining the magnification of the system. The values of ψ_2 are $-0.135\mu\text{m}$ and $0.508\mu\text{m}$ respectively.

The second order terms are significant in noting the differences in the aberrations between the parabolic and anti-symmetric rods. For the anti-symmetric rod the axial and distortion terms, ω_1 , ω_2 , and ω_6 , have approximately doubled and changed sign. The off-axis terms, ω_4 and ω_5 , although not zero at the full period, are considerably reduced. This is the opposite of the trend noted for the parabolic rod. The value of ω_3 remains zero as in the previous cases.

<u>Order</u>	<u>Coefficient*</u>	<u>$\frac{1}{2}$ Period</u>	<u>1 Period</u>
1st	ψ_1	0	0
	ψ_2	$-.135\mu\text{m}$	$.508\mu\text{m}$
	ψ_3	$-.012$	$.019$
2nd	ω_1	$-.190$	$.381$
	ω_2	$-.095$	$.190$
	ω_3	0	0
	ω_4	-1.547	$.168$
	ω_5	-1.525	$-.551$
	ω_6	$.034$	$-.067$
3rd	τ_1	$.141$	$-.279$
	τ_2	0	0
	τ_3	$.141$	$-.279$
	τ_4	0	0
	τ_5	0	0
	τ_6	0	0
	τ_7	0	0
	τ_8	$.118$	$-.238$
	τ_9	$-.394$	$.791$
	τ_{10}	0	0

*See Table 2.2 for a definition of the coefficients.

The Aberration Polynomial Coefficients for a
GRIN Rod Curved Along an Anti-Symmetric
Curve

Table 3.3

A comparison of the trends of the parabolic and anti-symmetric rods implies that the second order aberrations over a number of half periods is a summation of the aberrations of each period. The magnitude of the aberration is a function of the strength of the bend. The sign of the aberration in a specific half period is dependent upon the magnification and the sign of the slope of the bend. This dependence may be different for the various aberrations. This produces the canceling of the $\bar{p}^2$ and $\bar{h}^2$ terms, and the doubling of the $\bar{p}\bar{h}$ terms in the parabolic rod, and the opposite trend in the anti-symmetric rod. Further studies are required to determine the effect on the aberrations of the rod due to bending in a more quantified manner.

3.5 BENT ROD: PARABOLIC AND MODIFIED PROFILE

A significant imaging characteristic noted for the parabolic GD3 rod is the alternating of the axial image point off of and on to the physical axis at the half and full periods. Modifications to the index profile have been examined to determine a profile which causes the axial image point to coincide with the physical axis in all the image planes. The modification necessary is the introduction of a linear in Y term in ζ of the index expansion. This is the

gY term of Equation (2.18). The modified index profile of the GD3 rod curved along the parabolic curve given by Equation (3.2) is

$$n(\zeta) = N_{00} + N_{10}\zeta + N_{20}\zeta^2 , \quad (3.7)$$

where

$$\zeta = Y^2 + Z^2 - 0.267Y .$$

Figure 3.11 shows the imaging of an axial bundle of rays for the bent rod described by Equation (3.7). In this case the paraxial image plane is not normal to the axial ray within the tolerances specified within the program. The linear Y term added to the index profile causes the paraxial image plane to rotate about the axial image point. For this particular case the rotation is insignificant for the analysis of the aberration trends being performed. The rotation relative to the plane normal to the axial ray is less than 0.2° . To maintain the maximum accuracy possible, the paraxial image plane has been determined and the aberration analysis has been performed in this plane.

The values of the x_s' , y_s' and θ_s for the half period image are $-2.9 \mu\text{m}$, $0.83 \mu\text{m}$ and 0.06° respectively. At the full period image, the values are $-25.5 \mu\text{m}$, $1.0 \mu\text{m}$ and 0.13° .

Imaging the Axial Object Point for the
Parabolic Rod with a Modified
Index Profile
Figure 3.11

The aberration coefficients for the half and full period length rod are given in Table 3.4. A comparison with Table 3.2 shows the effect of the modification of the index profile on the aberrations. The third order coefficients are approximately the Seidel aberrations, and the ψ_3 term is within the acceptable limit. The value of the linear astigmatism, ψ_2 , has been reduced from the value for the unmodified rod. Some of the aberration terms have maintained the same trends when comparing the half period image to the full period. The third order aberrations have doubled and changed sign, as has been the case in all the rods that have been examined.

The second order distortion is the same in the half period image of both the modified and the unmodified rod and goes to zero at the full period image in both rods. The effect of the modified index profile is evident in the remaining second order terms. The off-axis aberrations, w_4 and w_5 , have doubled in the full period image but did not change signs. The axial aberrations, w_1 and w_2 , did not double or reduce to zero, but have become equal.

A comparison of the imaging quality of the modified and unmodified rods involves a series of trade-offs. A significant point is that the modified rod has no vignetting, whereas the unmodified rod does have vignetting

<u>Order</u>	<u>Coefficient*</u>	<u>$\frac{1}{2}$ Period</u>	<u>1 Period</u>
1st	ψ_1	0	0
	ψ_2	-.018 μ m	.022 μ m
	ψ_3	-.015 μ m	.031
2nd	ω_1	-.044	.022
	ω_2	-.026	.022
	ω_3	0	0
	ω_4	.032	.067
	ω_5	.033	.066
	ω_6	-.151	0
3rd	τ_1	.142	-.289
	τ_2	0	0
	τ_3	.142	-.288
	τ_4	0	0
	τ_5	0	0
	τ_6	0	0
	τ_7	0	0
	τ_8	.106	-.237
	τ_9	-.396	.791
	τ_{10}	0	0

*See Table 2.2 for a definition of the coefficients.

The Aberration Polynomial Coefficients for a
GRIN Rod With a Modified Index Profile
Curved Along a Parabolic Bend

Table 3.4

for all object heights. The total aberration at the various fractions of the field examined does not show general improvement or degradation for one case over the other. Some fractional field angles for the modified rod show improvement, but most have reduced imaging quality. Overall the vignetting is the key weighting factor; therefore the modification to the index profile has generally improved the imaging ability of the curved rod.

The parabolic curve considered in this example has only partial vignetting since the strength of the curve was limited to allow the comparison of the straight, bent, and modified rods. It is possible that a rod which is bent along a stronger curve, and therefore has total vignetting, could have the index profile modified in a manner similar to that described to allow imaging. The exact modification necessary would depend on the strength and type of curve introduced. The two linear in Y terms of Equation (2.18), gY and $2eYx$, could prove useful in this situation.

Chapter IV

CONCLUDING REMARKS

A raytracing program has been developed to perform the image analysis of bent gradient index rods. The model of the bent rod assumes that pure bending occurs during the transition from an equivalent straight rod to a curved rod. The effect on the optical properties due to bending of the rod is not included in the program, but a simple analysis of this effect indicates that the change in optical properties is significant in some cases.

A GRIN rod, with end-to-end imaging, bent along a parabolic curve is studied. The parabolic curve is selected to minimize the amount of vignetting. In the first two periods where the rod closely approximates the arc of a circle, a number of imaging characteristics are noted. The axial ray follows a path such that the distance from the physical axis of the rod varies as one minus the cosine of a function of the rods parameters. This results in the axial object point alternately imaging off of and onto the physical axis of the rod at the half and full period image planes. Another ray travels along a path that remains a

constant distance from the physical axis. The image plane shifts off the back endface of the rod and this focal shift is dependent upon the type and strength of the bend, and the length of the rod.

The coefficients of the transverse aberrations are analyzed in the $1/2$, 1, $1\ 1/2$, and 2 period image planes. The third order coefficients double and change sign with each half period. The bending of the rod introduces first and second order terms into the aberration polynomial. The first order, linear astigmatism increases with the number of half periods traversed. The off-axis second order coefficients vary in a manner similar to the third order. The second order axial coefficient has the same value for the $1/2$ and $1\ 1/2$ period length rods, and are zero for the 1 and 2 period length rods. The distortion coefficient exhibits the same behavior.

The transverse aberrations of a GRIN rod bent along an anti-symmetric curve are analyzed in the half and full period image planes. The third and first order aberrations follow the same trends as the parabolic rod. The second order follow an opposite trend, with the second order axial aberration and distortion coefficients doubling and the off-axis coefficients being significantly reduced.

Modifications of the radial gradient index profile are considered as a means of improving the image quality of a bent GRIN rod. The parabolic rod, with a term which is linear in Y added to the index profile, is analyzed. The transverse aberrations are examined at the half and full period image planes. Although the image quality at most object heights is reduced, there is no vignetting. Vignetting is the major problem when considering bent rods and the reduction, or elimination, of the vignetting is a significant accomplishment.

Further analysis of generally curved GRIN rods is necessary to more accurately determine the relationship between the even order aberrations and the bending. Studies of modifications of the index profile to allow imaging in a bent rod which otherwise is totally vignetted could be of interest. At present, the unavailability of gradient index materials with the proper index profile makes experimental analysis of bent GRIN rods impossible, except in the case of very mild bendings. Research in gradient index materials is on-going and future developments may make the experimental analysis of bent gradient index rods a possibility.

Appendix A

A LENS DECK

A lens deck for a radial gradient index rod bent along a parabolic curve is given in Figure A.1. The numbers in brackets are not part of the lens deck but are reference markers for this section. The input is similar to most lens design programs. The SURF card is used as the counter to set up the surface numbers of the optical system. The physical parameters of the elements are given between the appropriate SURF cards. The numbers on the SURF card indicate the type of media which follows, i.e., homogeneous or gradient, the curvature of the surface, the axial distance to the next surface, and the radius of the clear aperture.

The first surface [1] specified is the object which is on the endface of the rod. The index in object space is air, which is indicated by the "INDX 00" card. The second surface [2] is the front face of the rod, and the third [3] is the back endface. The physical parameters of the rod are listed after surface [2]. The base index is 2.4 and the N10 and N20 are given on the INDX 10 and 20 cards respectively.

	PARA		.375D0	8.3523D4	1.0D6	
[1]	SURF	1	0.0D0			
	INDX	00	1.0D0			
[2]	SURF	12	0.0D0	37.D0	.55D0	
	INDX	00	2.4D0			
	INDX	10	-.865122D-2			
	INDX	20	3.35051D-5			
	ANAM	11	1.0D0			
	ANAM	22	1.0D0			
	BNDY	2	4.8D-4			
[3]	SURF	1	0.0D0			
	SHAR		1.0D0			
	END					
	IMAG					
	MAGN					
	RAY		1.0D0	0.0D0	0.5D0	0.5D0
	YFAN		0.6D0			
	DIST					
	DONE					

A Typical Lens Deck

Figure A.1

The ANAM cards specify the values of the members of the Q matrix of Equation 2.25. The BNDY 2 card indicates the value of the a_2 coefficient of Equation 2.3 for the parabolic bend.

The user must also provide information on the first order properties of the straight rod. The PARA card lists the maximum object height, the radius of the entrance pupil, and the distance from the object to the entrance pupil. The SHAR card must also be included in the deck with the value of Δt , the extrapolation distance for the ray trace. The last card of the system specification section must be the END card.

A number of evaluation options are available. These options are briefly outlined as follows:

IMAG - determines the location of the image plane.

(See Figure 2.9)

MAGN - determines the paraxial magnification.

(See Figure 2.10)

RAY - traces a single ray at specified coordinates.

Determines the transverse aberrations by

FAN - tracing a fan of rays at a given fractional field angle and angle in the entrance pupil.

(See Figures 2.11-2.12)

DIST - traces a series of principle rays from a

fractional field angle to determine the distortion.

SPOT - generates a spot diagram with 25-100 rays

POLY - uses a least square method to fit the transverse aberration data to the aberration polynomial.

The IMAG and MAGN must be the first two cards of the evaluation section. The YFAN in Figure A.1 default the fan to a meridional fan from a fractional field angle of 0.6. The RAY card calls for a ray to be traced from the maximum object height, to the fractional entrance pupil coordinates of (0.5, 0.5). The last card of the lens deck must be the DONE card.

Appendix B

BEND-INDUCED INDEX GRADIENT

As a glass rod is flexed, the optical properties are altered. The change in the optical properties is dependent upon the amount of bending and the material properties of the base glass of the rod. Rather than attempt an extensive analysis a simplified model of the modifications to the optical properties is developed. The change in index in a GRIN rod due to bending can be broken into two contributions, the change in the homogeneous index and the change in the gradient. The contribution from the gradient index is considerably smaller than the homogeneous contributions, therefore, for an order of magnitude calculation, only the homogeneous contribution needs to be considered.

Figure 2.3b depicts the effect of pure bending on a segment of a glass rod as it is flexed. The shape of the segment can be approximated as resulting from an applied uniaxial pressure which varies linearly with distance from the neutral surface. The pressure is equal in magnitude, but opposite in sign, at equal distances above and below the

axis. This uniaxial pressure causes two effects which result in an index gradient. The pressure causes a stress within the rod which produces an induced-birefringence, a relative difference in the index in the directions parallel and perpendicular to the applied stress. The also pressure causes the tilting of the normal planes of the segment and thereby makes the volume, and density, of the glass a linear function of the distance from the neutral surface. The varying density within the rod results in an index gradient. The functional form of the index change due to these two effects is determined and the order of magnitude of each index gradient for some silicate glasses is calculated below.

The effect of uniaxial stress on glass and the resultant amount of induced birefringence was studied by Waxler and Napalitano [B.1]. The stress-optical coefficient, a materials constant which relates the birefringence to applied stress, was calculated from experimental data for a number of glasses. The relationship between the optical path difference, Δ , and the applied stress, σ , can be expressed as

$$\Delta = C_o \sigma d , \quad (B.1)$$

where d is the thickness of the glass sample, and C/d is the stress-optical coefficient. The relative change in index, Δn_b , is therefore given by

$$\Delta n_b = C_0 \sigma \quad . \quad (B.2)$$

The values of the stress-optical coefficient for various glasses can be found in a number of references [B.2, B.3, B.4]. Typical values of C_0 for silicate glasses range from $-1.5 \times 10^{-6} \text{ mm}^2/\text{N}$ to $3.5 \times 10^{-6} \text{ mm}^2/\text{N}$ at a wavelength of 589.3nm and room temperature.

The maximum amount of induced-birefringence for a bent rod will result when the rod is flexed to its fracture point. Bending beyond this point will cause the rod to break unless heat treatment of the glass is incorporated into the bending process. The heat treatment of glass requires a different analysis than the simplified model presented and will not be pursued. Doremus[B.5] gives the theoretical fracture stress of silicate glasses as $1.8 \times 10^{10} \text{ N/m}^2$. In practice, flaws in the surface of the glass, which concentrate applied stresses, reduce the actual fracture stress by a factor of 500 to 1000. A typical fracture stress is equal to $3.6 \times 10^7 \text{ N/m}^2$. Using this stress value induced birefringence at the edge of a homogeneous glass rod, flexed to its breaking point, is approximately

$$\Delta n_b = (36 \text{ N/mm}^2) C_o . \quad (\text{B.3})$$

The second effect on the index of the bent rod is due to the change in density of the glass at points off the neutral surface. A number of theoretical relationships relating the index of glass to its density have been developed. Three of these, by Drude[B.2], Lorentz-Lorenz[B.2], and Huggins, Sun and Davis[B.6], are examined and lead to similar approximations of the index change.

Both the Lorentz-Lorenz and Drude models consider the density of the glass to be a function of two independent variables, the volume and the temperature. The equation expressing these two theories is

$$\frac{dn_d}{n} = \phi (K_T dT + (K_V - 1) \frac{dV}{V}) , \quad (\text{B.4})$$

where

$$\phi = \frac{(n^2 - 1)}{2n} ,$$

$$\phi = \frac{(n^2 - 1)(n^2 + 2)}{6n^2}$$

for the Drude and Lorentz-Lorenz equation respectively, K_T is the temperature coefficient, and K_V is the volume coefficient. Since the flexing of the glass rod involves volume changes at constant temperatures, $dT = 0$, and the equation reduces to

$$\frac{dn_d}{n} = \phi (K_V - 1) \frac{dV}{V} , \quad (B.5)$$

where ϕ is as previously defined. Waxler and Weir[B.2] have reported on the effects of hydrostatic pressure on the refractive index of glass. In their experiments, the volume coefficient of many silicate glasses was determined to be 0.2, therefore Equation (B.5) for the case of silicate glass can be written as

$$\frac{dn_d}{n} = \phi (-0.8) \frac{dV}{V} . \quad (B.6)$$

The third relationship of the index gradient due to a density change was developed by Huggins, Sun and Davis. In this model the index of silicate glass is written as a function of its composition. The index is given by

$$n = 1 + \frac{R}{V} = 1 + \frac{\sum_i R_i}{\sum_i V_i} , \quad (B.7)$$

where R is the refraction per mole oxygen in the glass and V is the molar volume. The refraction and molar volume are both modeled as additive functions of the glass composition, resulting in the summations in Equation (B.7). As the glass rod is bent and the compression and tension are developed, as indicated in Figure 2.3b, the molar volume of the glass in the compressed region decreases, and increases in the area under tension. The volume change produces an index change which, to a first order approximation, can be determined by differentiating Equation (B.7) with respect to V ,

$$dn_d = \frac{-RdV}{V^2} . \quad (B.8)$$

Solving Equation (B.7) for R , and substituting into the above equation gives

$$dn_d = (1-n) \frac{dV}{V} . \quad (B.9)$$

The three equations relating the index change to a change in density have the same functional dependence on dV/V , but vary in the coefficient. The equations can be summarized as follows:

$$dn_d = \frac{KdV}{V} \quad (B.10)$$

where

$$K = (1-n)$$

$$K = \frac{-0.8 (n^2-1)}{2}$$

$$K = \frac{-0.8 (n^2-1) (n^2 + 2)}{6n}$$

for the Huggin, Sun, and Davis, Drude, and Lorentz-Lorenz models respectively. The K coefficients for the Lorentz-Lorenz and Drude models are in close agreement for the range of silicate glass indices. The K value for the Huggin, Sun and Davis model is not as close. An average of the three coefficients is used in all calculations of Δn_d .

To determine the value of dV/V for a glass rod at its fracture stress, it is necessary to make an assumption on the shape of the bend. For simplicity, the rod will be assumed to be bent along the arc of a circle. See Figure

B.1a. The intersection point of the normal planes of the rod is the center of the circle, labeled C. The variation in volume can be determined by examining Figure B.1b. Since pure bending in the Yx plane is assumed, the change in volume is due to the tilting of the normal planes, and therefore a change in the dimension ℓ . The nominal value of ℓ is ℓ_0 , the length along the neutral surface. The change in ℓ , $d\ell$, is equal to:

$$d\ell = \Delta = r\theta . \quad (B.11)$$

The value of ℓ_0 is

$$\ell_0 = R\theta . \quad (B.12)$$

The ratio of the volume change to the nominal volume is

$$\frac{dV}{V_0} = \frac{d\ell}{\ell} = \frac{r}{R} . \quad (B.13)$$

The value of $d\ell/\ell_0$, or r/R , for a rod flexed to its fracture stress must be calculated. A relationship between the applied stress in the rod and the value of R exists. The strain, ϵ , in a rod during pure bending is given by:

$$\epsilon = \frac{d\ell}{\ell} = \frac{r}{R} . \quad (B.14)$$

(b)

(a)

Bending a Rod along
the Arc of a Circle

Figure B.1

The strain, ϵ , is proportional to the stress, σ , within the glass. The constant of proportionality is Young's Modulus, E , and is a materials constant. The relationship relating stress and strain is

$$\sigma = E\epsilon = E \frac{r}{R} . \quad (B.15)$$

The equation for the change in index derived from the density change model is

$$dn_d = \frac{Kr}{R} = \frac{K\sigma}{E} . \quad (B.16)$$

Using the fracture point stress previously given, the maximum change in index under the present conditions is

$$\Delta n_d = (36 \text{ N/mm}^2) K/E \quad (B.17)$$

The order of magnitude of the total contribution to the index gradient induced by bending a rod at constant temperature is a combination of equations (B.3) and (B.17). The value of these changes in index have been calculated for some silicate glasses and are presented in Table B.1. The value of Δn_d is the total change in index from the neutral surface to the concave edge of the rod. The change in index to the convex edge is equal in magnitude, but opposite in sign. The value of Δn_b is negative indicating that the

<u>glass type</u>	<u>n</u>	<u>$(\times 10^{-6} \frac{C_o}{mm^2}/N)$</u>	<u>$(\times 10^3 \frac{E}{N/mm^2})$</u>	<u>K</u>	<u>Δn_d</u>	<u>Δn_b</u>
FK5	1.487	2.91	62	-.477	.00028	-.00011
FK51	1.487	0.67	79	-.476	.00022	-.00002
BK7	1.517	2.74	81	-.510	.00023	-.00010
ZKN7	1.509	3.62	71	-.500	.00025	-.00013
SK14	1.603	2.0	86	-.609	.00026	-.00007
SK16	1.620	1.9	89	-.630	.00025	-.00007
F2	1.620	2.81	58	-.629	.00039	-.00010
SF2	1.648	2.65	55	-.663	.00043	-.00010
SF4	1.755	1.35	56	-.797	.00051	-.00005
SF5	1.673	2.34	56	-.693	.00045	-.00008
SF6	1.805	0.63	56	-.862	.00055	-.00002
SF57	1.847	0.03	55	-.917	.00060	-.000001
SF59	1.953	-1.46	51	-1.065	.00075	-.00005

n= index of refraction

Δn_d = density gradient

C_o = stress-optical coefficient

E= Young's Modulus

Δn_b = birefringence

The change in index resulting from flexing a glass rod to its fracture point.

Table B.1

index gradient parallel to the applied stress is less than the gradient perpendicular. In all cases Δn_d is at least a factor of two times greater than Δn_p , and in most cases either value of index change would be considered a significant factor in the index profile of the rod.

Although the materials analysis present is not a complete one, the values calculated indicate that the effect on the index of refraction due to mild bendings may be significant. The simple model presented is adequate for the order of magnitude calculation performed. For a more accurate calculation, which would be necessary in an experimental study, a more complete model is required. The consideration of heat treatment affects on the rods for bendings beyond the breaking point would also be of interest. For the theoretical design and analysis of the image quality of curved rods, the origin of the index variation is not as significant a factor as the effect of this variation on the image. For this reason, the index variation due to bending the GRIN rod is not incorporated into the ray tracing program.

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