A Variant of Ben-Or's Lower Bound for Algebraic Decision Trees

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Summary. Ben-Or's lower bound on the height of a fixed-degree algebraic decision tree to decide membership in a set is based on the number of connected components in the set. A similar bound holds based on the sometimes much larger number of connected components in the interior of the set.

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Euclidean $n$-space $E^n$ is the set of all ordered $n$-tuples of real numbers. Within a subset $U$ of $E^n$, points $u$ and $v$ are connected if there is a continuous path $p: [0,1] \to E^n$ with $p(0) = u$, with $p(1) = v$, and with $p(t) \in U$ for every $t$ in the interval $[0,1] = \{ t \mid 0 \leq t \leq 1 \}$. Connectivity within $U$ is an equivalence relation, and its equivalence classes are the connected components of $U$. Let $\#(U)$ denote the number of connected components in $U$.

If $g(x_1, \ldots, x_n)$ is a multivariate polynomial of degree at most $d$, then it specifies a degree-$d$ test, the three possible outcomes of which are the degree-$d$ conditions $g(x_1, \ldots, x_n) < 0$, $g(x_1, \ldots, x_n) = 0$, and $g(x_1, \ldots, x_n) > 0$.

From a result of Milnor [7] and Thom [10], Ben-Or derived an upper bound of $d(2d-1)^{n+h-1}$ on the number of connected components in the set of points in $E^n$ satisfying a set of degree-$d$ conditions ($d \geq 2$), at most $h$ of which are inequalities [2, 8].

For classifying the points of $E^n$, we consider decision trees of height $h$, with 3-way decisions based on degree-$d$ tests. Corresponding to each node is the set of points in $E^n$ that satisfy the outcome conditions determining the path from the root to that node. Such a tree correctly decides membership in a set $W$ if each leaf's point set is included entirely in either $W$ or its complement.
Since a ternary decision tree of height $h$ can have at most $3^h$ leaves, Ben-Or's upper bound leads directly to a lower bound on $h$, following an earlier plan by Steele and Yao [9]:

**Ben-Or's Theorem** [2]. If membership in $W \subseteq E^n$ can be decided by a height-$h$ tree of degree-$d$ decisions ($d \geq 2$), then $\#(W) < 3^h d(2d-1)^{n+h-1}$, so that $$h \geq c_d (\log \#(W) - n),$$

where $c_d$ is some constant that depends only on $d$.

Ben-Or's theorem yields a good lower bound on the complexity of deciding membership in a set $W$ that has a large number of connected components. A good example is the set associated with the $n$-element distinctness problem: $\{(x_1, \ldots, x_n) \mid x_i$ and $x_j$ are distinct whenever $i$ and $j$ are$\}$. Since this set has a distinct connected component for each of $n!$ many order types, it requires depth proportional to $n \log n$ for each fixed degree.

A less fortunate example is the subset of $E^{2n}$ associated with the following yes-no version of the "convex hull" problem (see [8] for definitions): Given a set of $n$ points in the Euclidean plane $E^2$, determine whether or not all of them are extreme points on (i.e., vertices of) their convex hull. The origin in $E^{2d}$ is a "yes" point, and so are all the points on the straight-line segment joining each "yes" point to the origin; therefore, the "yes" region consists of a single connected component. The "no" region is connected, too; so Ben-Or's theorem does not come close to providing a nontrivial lower bound for this problem.

The inspiration for this note is the observation that the "yes" region's one connected component above actually consists of a large number of significant subcomponents, held together only by relatively insignificant "threads" of lower dimension. Unfortunately, however, a straightforward efficient reduction to the threadless problem does not seem possible, because recognizing the threads is essentially the element distinctness problem.

Our new contribution is a lower bound similar to Ben-Or's, but based on the number of connected components in the interior of $W$. (A point is in the interior $W^0$ of $W$ if all the points within some positive distance $\epsilon$ of the point lie within $W$. A set is open if all of its points are interior points.) The interior of the "yes" region above does have $(n-1)!$ connected components—one for each cyclic clockwise ordering of $n$ distinct extreme points on a convex hull in the plane.

**Theorem.** If membership in $W \subseteq E^n$ can be decided by a height-$h$ tree of degree-$d$ decisions ($d \geq 2$), then $\#(W^0) < 2^h d(2d-1)^{n+h-1}$, so that $$h \geq c_d (\log \#(W^0) - n),$$

where $c_d$ is some constant that depends only on $d$. 

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Proof: Consider any height- \( h \) tree of degree- \( d \) tests that classifies the points of \( E^n \) consistently with membership in \( W \). Without loss of generality, assume each test is nonconstant, since any constant test has only one possible outcome and can hence be eliminated. Call a leaf of this ternary tree open if the path to it from the root involves only inequalities. The set of points in \( E^n \) that lead to an open leaf is an open set, since it is a finite intersection of sets of the forms \( \{(x_1, \ldots, x_n) | g(x_1, \ldots, x_n) < 0\} \) and \( \{(x_1, \ldots, x_n) | g(x_1, \ldots, x_n) > 0\} \), which are open by continuity.

Note that each connected component of an open set is itself open, since any noninterior point would have a connected neighborhood entirely within the whole set but not entirely within the component, contradictorily providing a connection to another, supposedly different connected component.

The key to our proof is the observation that each connected component \( C_{W_0} \) of the open set \( W^0 \) must meet the open set corresponding to some open leaf. To see this, just consider the nonconstant polynomial \( p \) that is the product of all the tests in the tree. It is well known and easily shown by induction on \( n \) that no nonconstant polynomial function of \( n \) variables is constant on any open subset of \( E^n \). (The base case, \( n = 1 \), follows by familiar interpolation for univariate polynomials: A univariate polynomial of degree \( d \) is completely determined by its values at any \( d+1 \) points. The key to the induction step is the fact that each function obtained by fixing some of a polynomial's variables is a polynomial function of the remaining variables.) Therefore, since \( C_{W_0} \) is an open subset of \( E^n \), it must include a point that is not a zero for \( p \). Since such a point cannot be a zero for any factor of \( p \), it must lead to an open leaf.

In fact we can say more: Each connected component \( C_{W_0} \) of \( W^0 \) must entirely include a connected component of the set corresponding to some open leaf. To see this, let \( C_{\text{open leaf}} \) be any such connected component that \( C_{W_0} \) meets. Since the tree does correctly decide membership in \( W \), \( W \) must entirely include \( C_{\text{open leaf}} \). Since \( C_{\text{open leaf}} \) must be open, the interior \( W^0 \) must entirely include it. And since \( C_{\text{open leaf}} \) is connected, the connected component \( C_{W_0} \) that it meets must entirely include it.

We conclude that distinct connected components of \( W^0 \) must meet distinct connected components of the sets corresponding to open leaves. There can be at most \( 2^h \) open leaves, and each one's set can have at most \( d(2d-1)^{n+h-1} \) connected components (Ben-Or's upper bound); so the product is an upper bound on \( \#(W^0) \). \( \square \)

Comments on the Convex Hull Application

The lower bound we obtain for our yes-no version of the convex hull problem in the plane is proportional to \( n \log n \). To within a constant factor,
even the most ambitious version of the problem actually can be solved this fast [8]. Along with a number of solutions, lower bounds proportional to \( n \log n \) for some of the versions can be found in [8]; and Problem CH3 on page 95 there actually sounds like our version of the problem. The proof of the lower bound there, however, exploits an unstated assumption that distinctness is part of the question.

For degrees 1 and 2, lower bounds proportional to \( n \log n \) on exactly our problem are well known. The bound for degree 1 was obtained by van Emde Boas [3]. Although others soon observed that no number of degree-1 tests could suffice [1, 4], the insight provided by his focus on what we call "open leaves" was crucial to the development of our more general argument. The bound for degree 2 can be obtained by Yao's more difficult argument [11].

The bound for each degree greater than 2 is implicit in a result of Kirkpatrick and Seidel [6]. Their result is a lower bound proportional to \( n \log k \) on the height of any degree-\( d \) decision tree that correctly decides whether the convex hull of \( n \) distinct points has exactly \( k \) distinct extreme points. Our lower bound follows from the case \( k = n \). On the other hand, note that their result does not require the \textit{ad hoc} and relatively complicated proof they give—it is just another straightforward corollary of our variant of Ben-Or's theorem, since the interior of the entire subset of \( E^{2n} \) decided by any such tree is the same, and has \( (n - 1)(n - 2)\ldots(n - k + 1) \) connected components. Similarly, the analogous lower bounds in [5] for verifying the number of "maximal vectors" in a set are also straightforward corollaries of our new theorem.

These successes notwithstanding, there are conjectured lower bounds for variants of the convex hull problem that still seem to elude proof. For the variant that asks whether between one third and two thirds of the given \( n \) points are extreme, for example, both regions of \( E^{2n} \) seem very solidly connected. Also remaining open is Yao's conjecture that the height of a decision tree for the problem must be at least \( cn \log n \) for some constant \textit{that does not depend on the degree of the tests} [11].

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