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**SINGULARITIES OF EULER AND
ROLL-PITCH-YAW REPRESENTATIONS**

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ABSTRACT

Euler and Roll-Pitch-Yaw angles are routinely used to represent the orientation of rigid bodies in aerospace, navigation and robotics because they minimize the dimensionality of the control problem. Both representations, however, introduce unwarranted mathematical singularities which are identified in this paper. Trajectory-tracking algorithms break down at singularities and cause loss of control. Since mathematical singularities do not reflect physical limitations of orientation, remedial measures can be implemented in the controller.

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Figure 1: Definition of the reference and end-effector Cartesian coordinate frames.

1. INTRODUCTION

Kinematic models of articulated rigid bodies are needed in the analysis and control of many engineering systems. Satellites, robots, and mechanical linkages are a few examples of systems that provide for the controlled motion of rigid bodies in space to accomplish a variety of tasks [1]. The description of the position and orientation of a rigid body with respect to a Cartesian *reference* frame $\{R\} = \{O_R; \mathbf{x}_R, \mathbf{y}_R, \mathbf{z}_R\}$ is the subject of *kinematics*. In order to describe the motion of the rigid body, a Cartesian frame $\{E\} = \{O_E; \mathbf{x}_E, \mathbf{y}_E, \mathbf{z}_E\}$ is attached to it (Figure 1). At any time instant, i) the position of the rigid-body is defined by the (3×1) position vector $(O_R O_E)$ of the origin O_E of the rigid-body frame; and ii) the orientation of the rigid body is defined by the (3×3) rotation matrix $(\mathbf{x}_E, \mathbf{y}_E, \mathbf{z}_E)$ where the *unit* vectors $\mathbf{x}_E$, $\mathbf{y}_E$ and $\mathbf{z}_E$ describe the axes of the rigid-body frame:

$$(O_R O_E) = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}; \quad (\mathbf{x}_E, \mathbf{y}_E, \mathbf{z}_E) = \begin{pmatrix} n_x & o_x & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{pmatrix}.$$

(All vectors are expressed in the reference frame.)

In this paper, we focus upon mathematical representations of the orientation of a rigid body. (We assume, therefore, that $O_E \equiv O_R \Leftrightarrow p_x = p_y = p_z = 0$.) The rotation matrix approach utilizes nine parameters (which obey orthogonality and unit length constraints) to describe the orientation of a rigid body. Alternative orientation representations utilize a quadruple of ordered real parameters $(s; k_x, k_y, k_z)$. Orientation is then defined by a finite rotation s about the axis $\mathbf{k} = (k_x, k_y, k_z)^T$ described in the reference frame. *Quaternions* are a special case of four-parameter representations that have been applied to the analysis of spatial mechanisms [2]. A quaternion representation is defined by the entity $\{\cos \frac{s}{2} + \sin \frac{s}{2} \mathbf{k}\}$. Four-parameter representations are related to the elements of the rotation matrix and inverse transformations exist to compute these parameters from a given rotation matrix. Since a rigid body possesses three rotational degrees of freedom, three independent parameters should be enough to characterize completely and unambiguously its orientation. Three-parameter representations are popular in engineering because they minimize the dimensionality of the rigid-body control problem [1]. The most commonly used representations of orientation are the *Euler* angles (α, β, γ) and the *Roll-Pitch-Yaw* angles (ϕ, θ, ψ) . The Euler and Roll-Pitch-Yaw angles are related to the elements of the rotation matrix representation and an *inverse* transformation is routinely used to compute these orientation angles from a given rotation matrix [3].

Mathematical singularities occur at the intersections between multiple solution regions to the inverse transformation problem. These singularities signify orientations in space where infinite rotational rates are required to produce a finite angular velocity and acceleration of the rigid body [3]. Mathematical singularities may arise depending on the representation used for orientation and do *not* reflect physical limitations of the rigid body. The inverse transformation algorithm breaks down at singular orientations, and hence mathematical singularities must be avoided during task planning [4]. Orientation representations that introduce mathematical singularities are deficient and should be carefully analyzed. The objective of this paper is i) to identify the mathematical singularities associated with the use of Euler and Roll-Pitch-Yaw angles; ii) provide physical insight into the orientations they represent; and iii) introduce remedial measures in the control algorithm.

The paper is organized as follows. In Sections 2 and 3, we introduce the Euler-

angle and Roll-Pitch-Yaw representations respectively. In Section 4, we describe the orientation control problem and, in Section 5, we present an example from robotics to illustrate the development of the paper. Finally, in Section 6, we summarize our results and identify areas for future research.

2. THE EULER REPRESENTATION

In the Euler representation, a series of three ordered *right-handed* rotations is needed to coalesce the reference frame with the rigid-body frame. Specifically, the reference frame is rotated: i) about the $\mathbf{z}_R = [0 \ 0 \ 1]^T$ axis by an angle α , so that the plane $(\mathbf{x}_R, \mathbf{z}_R)$ contains $\mathbf{z}_E$; then ii) about the new $\mathbf{y}_R = [-\sin \alpha \ \cos \alpha \ 0]^T$ axis by an angle β so that $\mathbf{z}_R \equiv \mathbf{z}_E$; and finally iii) about the new $\mathbf{z}_R = [\cos \alpha \sin \beta \ \sin \alpha \sin \beta \ \cos \beta]^T$ axis by an angle γ so that $\mathbf{x}_R \equiv \mathbf{x}_E$. We illustrate the definition of the Euler angles in Figure 2. (For expository convenience, we show only the pertinent axes at each step.) The three aforementioned rotations justify the following rotation matrix $(\mathbf{x}_E, \mathbf{y}_E, \mathbf{z}_E)$ representation:

$$\begin{aligned} Euler(\alpha, \beta, \gamma) &= (\mathbf{x}_E \ \mathbf{y}_E \ \mathbf{z}_E) \\ &= \begin{pmatrix} \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma & -\cos \alpha \cos \beta \sin \gamma - \sin \alpha \cos \gamma & \cos \alpha \sin \beta \\ \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma & -\sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \alpha \sin \beta \\ -\sin \beta \cos \gamma & \sin \beta \sin \gamma & \cos \beta \end{pmatrix}. \end{aligned}$$

The orientation of the rigid-body is thus completely specified by the three Euler angles α, β, γ .

The inverse transformation can be computed readily from the rotation matrix by observing that:

$$\begin{aligned} a_x &= \cos \alpha \sin \beta & n_z &= -\sin \beta \cos \gamma \\ a_y &= \sin \alpha \sin \beta & o_z &= \sin \beta \sin \gamma \\ a_z &= \cos \beta \end{aligned}$$

If $\sin \beta \neq 0 \iff |a_x| + |a_y| \neq 0$, the inverse transformation problem has two solutions (α, β, γ) and $(\alpha + 180^\circ, -\beta, \gamma + 180^\circ)$ where:

$$\alpha = \text{atan2}(a_y, a_x); \quad \beta = \text{atan2}\left(+\sqrt{a_x^2 + a_y^2}, a_z\right); \quad \gamma = \text{atan2}(o_z, -n_z).$$

A mathematical singularity occurs at the boundary of the two solutions, $\sin \beta = 0 \iff$

Figure 2: Euler Angles Definition.

Figure 3: Singular Orientation (Euler Angles).

$a_x = a_y = 0$:

$$\begin{pmatrix} n_x & o_x & 0 \\ n_y & o_y & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \Leftrightarrow \begin{cases} \beta = 0^\circ, & \gamma + \alpha = \text{atan2}(n_y, n_x) \\ \beta = 180^\circ, & \gamma - \alpha = \text{atan2}(n_y, -n_x) \end{cases}$$

At this singular orientation, $\mathbf{z}_E \parallel \mathbf{z}_R$ ($a_z = 1 \Leftrightarrow \beta = 0^\circ$ or $a_z = -1 \Leftrightarrow \beta = 180^\circ$); α and γ describe the same rotation and cannot be computed separately; and one degree of freedom is lost. We depict the singularity $\beta = 0^\circ$ in Figure 3.

When $\beta = 0^\circ$ or 180° , infinite rotational rates $(\dot{\alpha}, \dot{\beta}, \dot{\gamma})$ are required to produce a finite rigid-body angular velocity. Indeed, from the definition of the Euler angles, the angular velocity of the rigid-body $\mathbf{w} = [\omega_x, \omega_y, \omega_z]^T$ can be computed as

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{\alpha} + \begin{pmatrix} -\sin \alpha \\ \cos \alpha \\ 0 \end{pmatrix} \dot{\beta} + \underbrace{\begin{pmatrix} \cos \alpha \sin \beta \\ \sin \alpha \sin \beta \\ \cos \beta \end{pmatrix}}_{\mathbf{J}_{Euler}} \dot{\gamma} = \begin{pmatrix} 0 & -\sin \alpha & \cos \alpha \sin \beta \\ 0 & \cos \alpha & \sin \alpha \sin \beta \\ 1 & 0 & \cos \beta \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix}.$$

The mathematical singularity, $\beta = 0^\circ$ or 180° , characterizes the orientation where the determinant of the Jacobian matrix $\det[\mathbf{J}_{Euler}] = -\sin \beta$ vanishes.

The angular acceleration of the rigid-body $\dot{\mathbf{w}} = [\dot{\omega}_x, \dot{\omega}_y, \dot{\omega}_z]^T$ is computed by differentiating the angular velocity with respect to time:

$$\dot{\mathbf{w}} = \begin{pmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{pmatrix} = \mathbf{J}_{Euler} \begin{pmatrix} \ddot{\alpha} \\ \ddot{\beta} \\ \ddot{\gamma} \end{pmatrix} + \underbrace{\begin{pmatrix} -\cos \alpha & -\sin \alpha \sin \beta & \cos \alpha \cos \beta \\ -\sin \alpha & \cos \alpha \sin \beta & \sin \alpha \cos \beta \\ 0 & 0 & -\sin \beta \end{pmatrix}}_{\mathbf{K}_{Euler}} \begin{pmatrix} \dot{\alpha} \dot{\beta} \\ \dot{\alpha} \dot{\gamma} \\ \dot{\beta} \dot{\gamma} \end{pmatrix}.$$

Both the angular velocities $(\dot{\alpha}, \dot{\beta}, \dot{\gamma})$ and angular accelerations $(\ddot{\alpha}, \ddot{\beta}, \ddot{\gamma})$ contribute to the angular acceleration $(\dot{\mathbf{w}})$ of the rigid-body. At the singular orientation, $\det[\mathbf{J}_{Euler}] = 0$ and infinite angular accelerations are needed to produce a finite rigid-body angular acceleration. (We note that $\det[\mathbf{K}_{Euler}] = \sin^2 \beta$, and if $\mathbf{J}_{Euler}$ is singular, $\mathbf{K}_{Euler}$ is also singular.)

3. THE ROLL-PITCH-YAW REPRESENTATION

In the Roll-Pitch-Yaw representation, a series of three ordered *right-handed* rotations is needed to coalesce the reference frame with the rigid-body frame. Specifically, the reference frame is rotated: i) about the $\mathbf{z}_R = [0 \ 0 \ 1]^T$ axis by an angle ϕ , so that the plane $(\mathbf{x}_R, \mathbf{z}_R)$ contains $\mathbf{x}_E$; then ii) about the new $\mathbf{y}_R = [-\sin \phi \ \cos \phi \ 0]^T$ axis by an angle θ so that $\mathbf{x}_R \equiv \mathbf{x}_E$; and finally iii) about the new $\mathbf{x}_R = [\cos \phi \cos \theta \ \sin \phi \cos \theta \ -\sin \theta]^T$ axis by an angle ψ so that $\mathbf{z}_R \equiv \mathbf{z}_E$. We illustrate the definition of the Roll-Pitch-Yaw angles in Figure 4. (For expository convenience, we show only the pertinent axes at each step.) The three aforementioned rotations justify the following rotation matrix $(\mathbf{x}_E, \mathbf{y}_E, \mathbf{z}_E)$ representation:

$$\begin{aligned} RPY(\phi, \theta, \psi) &= (\mathbf{x}_E \ \mathbf{y}_E \ \mathbf{z}_E) \\ &= \begin{pmatrix} \cos \phi \cos \theta & \cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi & \cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi \\ \sin \phi \cos \theta & \sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi & \sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi \\ -\sin \theta & \cos \theta \sin \psi & \cos \theta \cos \psi \end{pmatrix}. \end{aligned}$$

The orientation of the rigid-body is thus completely specified by the three Roll-Pitch-Yaw angles ϕ, θ, ψ .

The inverse transformation can be computed readily from the rotation matrix by observing that:

$$\begin{aligned} n_x &= \cos \phi \cos \theta & o_z &= \cos \theta \sin \psi \\ n_y &= \sin \phi \cos \theta & a_z &= \cos \theta \cos \psi \\ n_z &= -\sin \theta \end{aligned}$$

Figure 4: Roll-Pitch-Yaw Angles Definition.

Figure 5: Singular Orientation (Roll-Pitch-Yaw Angles).

If $\cos \theta \neq 0 \iff |n_x| + |n_y| \neq 0$, the inverse transformation problem has two solutions (ϕ, θ, ψ) and $(\phi + 180^\circ, 180^\circ - \theta, \psi + 180^\circ)$ where:

$$\phi = \text{atan2}(n_y, n_x); \quad \theta = \text{atan2}(-n_z, \sqrt{n_x^2 + n_y^2}); \quad \psi = \text{atan2}(o_z, a_z).$$

A mathematical singularity occurs at the boundary of the two solutions, $\cos \theta = 0 \iff n_x = n_y = 0$:

$$\begin{pmatrix} 0 & o_x & a_x \\ 0 & o_y & a_y \\ \pm 1 & 0 & 0 \end{pmatrix} \Leftrightarrow \begin{cases} \theta = 90^\circ, & \psi - \phi = \text{atan2}(o_x, o_y) \\ \theta = 270^\circ, & \psi + \phi = \text{atan2}(-o_x, o_y) \end{cases}.$$

At this singular orientation, $\mathbf{z}_E \perp \mathbf{z}_R$ ($n_z = 1 \Leftrightarrow \theta = 270^\circ$ or $n_z = -1 \Leftrightarrow \theta = 90^\circ$); ϕ and ψ describe the same rotation and cannot be computed separately; and one degree of freedom is lost. We depict the singularity $\theta = 270^\circ$ in Figure 5.

When $\theta = 90^\circ$ or 270° , infinite rotational rates $(\dot{\phi}, \dot{\theta}, \dot{\psi})$ are required to produce a finite rigid-body angular velocity. Indeed, from the definition of the Roll-Pitch-Yaw

angles, the angular velocity of the rigid-body $\mathbf{w} = [\omega_x, \omega_y, \omega_z]^T$ can be computed as

$$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \dot{\phi} + \begin{pmatrix} -\sin \phi \\ \cos \phi \\ 0 \end{pmatrix} \dot{\theta} + \underbrace{\begin{pmatrix} \cos \phi \cos \theta \\ \sin \phi \cos \theta \\ -\sin \theta \end{pmatrix}}_{\mathbf{J}_{RPY}} \dot{\psi} = \begin{pmatrix} 0 & -\sin \phi & \cos \phi \cos \theta \\ 0 & \cos \phi & \sin \phi \cos \theta \\ 1 & 0 & -\sin \theta \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}.$$

The mathematical singularity, $\theta = 90^\circ$ or 270° , characterizes the orientation where the determinant of the Jacobian matrix $\det[\mathbf{J}_{RPY}] = -\cos \theta$ vanishes.

The angular acceleration of the end-effector $\dot{\mathbf{w}} = [\dot{\omega}_x, \dot{\omega}_y, \dot{\omega}_z]^T$ is computed by differentiating the angular velocity with respect to time:

$$\dot{\mathbf{w}} = \begin{pmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{pmatrix} = \mathbf{J}_{RPY} \begin{pmatrix} \ddot{\phi} \\ \ddot{\theta} \\ \ddot{\psi} \end{pmatrix} + \underbrace{\begin{pmatrix} -\cos \phi & -\sin \phi \cos \theta & -\cos \phi \sin \theta \\ -\sin \phi & \cos \phi \cos \theta & -\sin \phi \sin \theta \\ 0 & 0 & -\cos \theta \end{pmatrix}}_{\mathbf{K}_{RPY}} \begin{pmatrix} \dot{\phi} \dot{\theta} \\ \dot{\phi} \dot{\psi} \\ \dot{\theta} \dot{\psi} \end{pmatrix}.$$

Both the angular velocities $(\dot{\phi}, \dot{\theta}, \dot{\psi})$ and angular accelerations $(\ddot{\phi}, \ddot{\theta}, \ddot{\psi})$ contribute to the angular acceleration $(\dot{\mathbf{w}})$ of the rigid-body. At the singular orientation, $\det[\mathbf{J}_{RPY}] = 0$ and infinite angular accelerations are needed to produce a finite rigid-body angular acceleration. (We note that $\det[\mathbf{K}_{RPY}] = \cos^2 \theta$, and if $\mathbf{J}_{RPY}$ is singular, $\mathbf{K}_{RPY}$ is also singular.)

4. THE ORIENTATION CONTROL PROBLEM

The orientation control problem centers around the computation of the actuating torques which are required to produce the rigid-body motion. Effective control is achieved by means of reducing the orientation error to zero. The orientation control problem is complicated by the practical fact that the desired motion is described in the reference coordinate frame rather than in the natural Euler (or Roll-Pitch-Yaw) coordinates of the orienting mechanism. Since the joints of the orienting mechanism define the axes around which motions of accommodation can occur, the controller must resolve the rigid-body motion in the Euler (or Roll-Pitch-Yaw) coordinates.

Most commercial orienting mechanisms are equipped with conventional PID controllers which are based upon simplified linear models of the joints. These simplified models fail to characterize the complex joint dynamics and result in oscillations or overshoot. The complexity of the highly coupled and nonlinear second-order differential equations that describe the dynamics of orienting mechanisms has led to the

development of model-based control algorithms for orientation [5,6]. These nonlinear feedback control algorithms utilize the full dynamic model to control the joints of the orienting mechanism.

A schematic representation of the orientation control problem appears in Figure 6. The desired motion is defined by a sequence of unit vectors, $(\mathbf{n}_d, \mathbf{o}_d, \mathbf{a}_d)$, which describe the axes $(\mathbf{x}_E, \mathbf{y}_E, \mathbf{z}_E)$ of the rigid-body frame in the reference frame. The error evaluator defines the error between the desired and actual orientations as [5]:

$$\boldsymbol{\epsilon}(t) = \frac{1}{2}[(\mathbf{n} \times \mathbf{n}_d) + (\mathbf{o} \times \mathbf{o}_d) + (\mathbf{a} \times \mathbf{a}_d)].$$

The orientation error is then resolved into the (3×3) joint error vector $\mathbf{e}(t)$ through an inverse Jacobian transformation:

$$\boldsymbol{\epsilon}(t) = \mathbf{J}\mathbf{e}(t) \iff \mathbf{e}(t) = \mathbf{J}^{-1}\boldsymbol{\epsilon}(t).$$

(Each joint of the orienting mechanism is assumed to be actuated independently and the coordinated motion of the joints results in the relative orientation of the rigid body; i.e., $\mathbf{q} = [\alpha, \beta, \gamma]^T$, or $\mathbf{q} = [\phi, \theta, \psi]^T$, and $\mathbf{e}(t) = \mathbf{q}_d - \mathbf{q}$.) In order to reduce the orientation error, the control signal $\mathbf{u}(t)$ incorporates both joint (angular) position and velocity feedback:

$$\mathbf{u}(t) = -\mathbf{K}_v\dot{\mathbf{q}} + \mathbf{K}_p\mathbf{e}$$

where $\mathbf{K}_p$ and $\mathbf{K}_v$ are the (3×3) constant position and velocity feedback gain matrices, respectively. The controller utilizes the dynamic model of the orienting mechanism and the actuators to generate the driving set of torques

$$\mathbf{F}(t) = \mathbf{D}(\mathbf{q})\mathbf{u}(t) + \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}).$$

The dynamics of the orienting mechanism are described by the (3×3) inertial matrix $\mathbf{D}(\mathbf{q})$ and by the (3×1) coupling vector $\mathbf{h}(\mathbf{q}, \dot{\mathbf{q}})$ [3,6].

When the rigid body moves *through* a singular orientation, the inverse transformation $\mathbf{e}(t) \mapsto \boldsymbol{\epsilon}(t)$ breaks down and the control algorithm collapses. This is due to the fact that the determinant of the Jacobian matrix $\mathbf{J}$ vanishes at the singularity. When the rigid body moves *in the neighborhood of* a singular orientation, the determinant of the Jacobian matrix is very small. In this case, the magnitudes of the joint error $\mathbf{e}(t)$, the control signal $\mathbf{u}(t)$ and the driving torque $\mathbf{F}(t)$ increase sharply. While the

Figure 6: The Orientation Control Problem.

mathematical algorithm is still operational, the actuators of the orienting mechanism become saturated and there is momentary loss of control.

As a remedy for this deficiency of the Euler and Roll–Pitch–Yaw orientation representations, we propose the use of a saturation nonlinearity:

$$F(t) = \begin{cases} F_{sat}, & \text{for } F(t) > F_{sat} \\ F(t), & \text{for } -F_{sat} \leq F(t) \leq F_{sat} \\ -F_{sat}, & \text{for } F(t) < -F_{sat} \end{cases}$$

where F_{sat} is the maximum torque that can be provided by the actuators. The saturation nonlinearity will *clip* the driving torques as soon as they exceed their nominal values in order not to damage the actuators. This conservative approach safeguards against a numerical breakdown of the controller and allows the system to recover at the expense of increasing the joint errors locally. We demonstrate these concepts through an example in the next section.

5. AN EXAMPLE FROM ROBOTICS

In this section, we highlight the development of the paper by designing a controller for the spherical wrist (*orienting mechanism*) of the JPL robot arm [7]. The wrist, depicted schematically in Figure 7, consists of three links connected by three independently actuated rotary joints. The axes of rotation are $\mathbf{z}_0, \mathbf{z}_1$ and $\mathbf{z}_2$ respectively. To describe the geometry of the wrist mechanism, we assign a coordinate frame $(\mathbf{O}_i; \mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ to link i , where $i \in \{0, 1, 2, 3\}$, according to the Denavit-Hartenberg convention [3]:

- The $\mathbf{z}_i$ axis is directed along the axis of motion of joint $(i + 1)$;
- The $\mathbf{x}_i$ axis lies along the common normal from the $\mathbf{z}_{i-1}$ axis to the $\mathbf{z}_i$ axis; i.e.,

$$\mathbf{x}_i = \frac{\mathbf{z}_{i-1} \times \mathbf{z}_i}{\|\mathbf{z}_{i-1} \times \mathbf{z}_i\|}$$

(if $\mathbf{z}_{i-1} \times \mathbf{z}_i = 0$, then $\mathbf{x}_i$ is arbitrary, subject only to $\mathbf{x}_i^T \mathbf{z}_i = 0$); and

- The $\mathbf{y}_i$ axis completes the right-handed coordinate system; i.e., $\mathbf{y}_i = \mathbf{z}_i \times \mathbf{x}_i$.

The zero frame serves as the reference frame; i.e., $(\mathbf{O}_R; \mathbf{x}_R, \mathbf{y}_R, \mathbf{z}_R) \equiv (\mathbf{O}_0; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$, and the third frame is the rigid body frame; i.e., $(\mathbf{O}_E; \mathbf{x}_E, \mathbf{y}_E, \mathbf{z}_E) \equiv (\mathbf{O}_3; \mathbf{x}_3, \mathbf{y}_3, \mathbf{z}_3)$.

$$O_R = O_0 = O_1 = O_2$$

$$O_E = O_3$$

Figure 7: JPL Spherical Wrist (Singular Orientation).

<i>Kinematic Parameters</i>				
<i>Link</i>	θ	r	d	a
1	α	0	0	-90°
2	β	0	0	90°
3	γ	0.2476	0	0

Table 1: JPL Spherical Wrist (Kinematics).

(The kinematic parameters that relate the position and orientation of the link coordinate frames are summarized in Table 1.) Each joint coordinate q_i , $i \in \{1, 2, 3\}$, is defined as the angular displacement around $\mathbf{z}_{i-1}$. The ordered sequence of joint rotations physically imitates the Euler angles that describe the orientation of the rigid body; i.e., $\mathbf{q} = [q_1, q_2, q_3]^T = [\alpha, \beta, \gamma]^T$.

The spherical wrist has a physical singularity which corresponds to the mathematical singularity of the Euler representation. This singularity occurs when the second joint is at $\beta = 0^\circ$ or $\beta = 180^\circ$. At this orientation, $\mathbf{z}_0$ is aligned with $\mathbf{z}_2$ and one rotational degree of freedom is lost. (The wrist is depicted in Figure 7 at the singular orientation.) To identify the effects of this singularity on the controller, we designed two simple simulation experiments that require the wrist to pass near the singular orientation. Specifically, the wrist is required to move the rigid body from its initial orientation

$$(\mathbf{n}, \mathbf{o}, \mathbf{a}) = \begin{pmatrix} 0.7392 & -0.2803 & 0.6124 \\ 0.5732 & 0.7392 & -0.3536 \\ -0.3536 & 0.6124 & 0.7071 \end{pmatrix} \iff (\alpha, \beta, \gamma) = (-30^\circ, 45^\circ, 60^\circ)$$

to the following desired orientation:

$$(\mathbf{n}_d, \mathbf{o}_d, \mathbf{a}_d) = \begin{pmatrix} -0.3420 & -0.9254 & -0.1632 \\ 0.9397 & -0.3368 & -0.0594 \\ 0.0000 & -0.1736 & 0.9848 \end{pmatrix} \iff (\alpha_d, \beta_d, \gamma_d) = (20^\circ, -10^\circ, 90^\circ).$$

Dynamic Parameters							
Link	Mass	$\bar{x}$	$\bar{y}$	$\bar{z}$	I_{xx}	I_{yy}	I_{zz}
1	1.08	0	0.0092	-0.0054	0.002	0.001	0.001
2	0.63	0	0	0.0566	0.003	0.003	0.0004
3	0.51	0	0	-0.0922	0.005	0.005	0.0003

Table 2: JPL Spherical Wrist (Dynamics).

The rigid body is initially at rest and it has to reach the final orientation with zero velocity (*point-to-point control*).

In our simulation experiments, we implemented a model-based controller to compute the necessary driving torques:

$$\mathbf{F}(t) = \mathbf{D}(\mathbf{q})\mathbf{u}(t) + \mathbf{h}(\mathbf{q})$$

where

$$\mathbf{u}(t) = -k_v \dot{\mathbf{q}} + k_p \mathbf{J}_{Euler}^{-1} \boldsymbol{\epsilon} = -k_v \dot{\mathbf{q}} + k_p \mathbf{e}$$

and $\mathbf{h}(\mathbf{q})$ is a feedforward compensation for gravity. (Centrifugal and Coriolis terms are neglected in the controller.) The dynamic parameters of the links are summarized in Table 2.

We selected *diagonal* position and velocity gain matrices to decouple the Euler angles. For expository convenience, we specified $\mathbf{K}_p = k_p \mathbf{I}$ and $\mathbf{K}_v = k_v \mathbf{I}$ where k_p and k_v are identical scalar feedback gains and $\mathbf{I}$ is the (3×3) identity matrix. (In practice, the designer can select different position and velocity feedback gains for each joint.) We then implemented the *critically-damped* design

$$k_v^2 = 4k_p \Leftrightarrow k_v = 2\lambda > 0 \text{ and } k_p = \lambda^2$$

which provides the fastest possible response without overshoot [8]. (It should be noted that in the presence of modeling errors, the Euler angles are *not* uncoupled and their time response is *not* necessarily critically-damped.) We chose $k_p = 100s^{-2}$ and $k_v = 20s^{-1}$ to specify a pair of closed-loop eigenvalues at $s = -10s^{-1}$.

We display the angular errors and the driving torques of the first experiment in Figures 8 and 9 respectively. We observe that near the singular orientation ($\beta \cong 0.003^\circ$ at $t = 0.414$ s) the magnitudes of the angular errors ($\alpha_d - \alpha$) and ($\gamma_d - \gamma$) sharply rise and there is a dramatic increase in the torque required to bring the system under control ($F_2 \cong 86Nm$ at $t = 0.414$ s). The peak torque load is more than 40 times the normal operating range of 0–2 Nm for this motion and cannot be handled by the actuators.

In the second experiment, we redesigned our algorithm to include a saturation nonlinearity in the controller. We display the angular errors and the driving torques in Figures 10 and 11 respectively. In this case, the torque F_2 is kept within the operating range (as anticipated) and the angular errors ($\alpha_d - \alpha$) and ($\gamma_d - \gamma$) exhibit a slight deviation from a critically damped response at $t = 0.414$ s.

6. CONCLUSIONS

We have highlighted the Euler and Roll-Pitch-Yaw representations for orientation and presented the forward and inverse transformations that relate them to the rotation matrix approach. Both representations utilize a minimal set of three parameters to describe orientation and are thus appealing from the control point of view. We have shown, however, that the Euler and Roll-Pitch-Yaw representations are deficient because they introduce unwarranted mathematical singularities. Care, therefore, should be exercised in using these representations for tracking applications and remedial measures must be included in the control algorithm.

Future research efforts should focus upon two parallel objectives: (i) the development of three-parameter representations for orientation that do *not* exhibit mathematical singularities; and (ii) the integration of singularity compensators in the orientation control system design.

Figure 8: Torque Profiles for Experiment I.

Figure 9: Angular Errors for Experiment I.

Figure 10: Torque Profiles for Experiment II.

Figure 11: Angular Errors for Experiment II.

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