TM-58

**ANALYSIS AND DESIGN OF
ROBOTIC MANIUPATORS WITH
MULTIPLE INTERCHANGEABLE WRISTS**

by

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PRODUCTION AUTOMATION PROJECT

March 1987

Technical Memorandum No. 58

PRODUCTION AUTOMATION PROJECT

College of Engineering & Applied Science

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Rochester, New York 14627

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Submitted to the *IEEE Journal of Robotics & Automation*

The work described in this report is supported by the Department of Electrical Engineering and by companies in the P.A.P.'s Industrial Associates Program. Any opinions, findings, conclusions, or recommendations expressed in this report are those of the authors and do not necessarily reflect the views of the industrial sponsors or the University of Rochester.

ABSTRACT

Robotic applications are constrained by the type of wrist a robot uses. The concept of multiple *interchangeable* wrist mechanisms is introduced in this paper as a cost-effective, practical method to improve the flexibility of a robot system without increasing the complexity of the wrist design. The interchangeable wrist concept is a natural extension of interchangeable tooling. The objective of this paper is to determine the feasibility of the proposed concept for *real-time* applications. An analytical framework that incorporates the wrist-exchange procedure in the robot control hierarchy is presented for the case of manipulators with interchangeable spherical wrists. An algorithmic approach that utilizes the individual kinematic properties of the arm and the wrists is then developed to generate an efficient solution to the kinematic control problem of the coupled system.

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1 Introduction

Industrial manipulators are programmed to interact with the environment by positioning and orienting an *end-effector* in space. Typical six degree-of-freedom robots consist of two distinct, three degree-of-freedom subsystems: the positioning *arm* and the orienting *wrist*. The arm is primarily responsible for positioning the wrist at a specified point while the wrist is responsible for orienting the end-effector. Although the two subsystems have different tasks, attaching the wrist to the arm makes the manipulator a highly coupled system. Motion of the arm may not preserve the orientation of the wrist, while motion of the wrist may alter the position of the end-effector. The robotic capability for controlled motion depends upon the *coordination* of the two subsystems [1].

Robotic applications are constrained by the type of wrist a robot uses. Wrist designs have been traditionally customized for specific tasks and operating conditions. This conventional approach capitalized on the robot's ability to learn repetitive tasks and execute them over a long period of time. As product life-cycles continue to shorten, there is a growing need for increased robotic flexibility and cost justification. (If a robot can perform multiple tasks within a single work station, more value will be added to the material by shortening the load/unload cycles.) The need can be addressed by designing sophisticated wrist mechanisms that can handle large families of tasks. Research efforts to emulate the human hand have produced *universal* wrists with individually articulated fingers that, in principle, can hold and use ordinary tools [2]. These multifunctional wrists, however, involve significant development costs and lack the dynamic stiffness to permit operations such as machining without chatter. Furthermore, the use of multifunctional wrists may not be appropriate for *teleoperators* employed in environments hazardous or inaccessible to humans, such as deep-sea mining, nuclear energy production facilities or space [3]. A teleoperator is a manipulator that requires the supervision of a human operator from a remote location [4]. The fine-motion capabil-

ity of a teleoperator is governed by the rate of information transmission between the manipulator and the operator [5]. The use of universal wrists can decrease dramatically the information transmission rate and lead to unacceptable performance.

In this paper, we introduce the concept of *interchangeable* wrist mechanisms as a cost-effective, practical method to improve the flexibility of the robot system without increasing the complexity of the wrist design. The interchangeable wrist concept is a natural extension of interchangeable tooling [6, Chapter 5] and is schematically depicted in Figure 1. The individual wrists are mounted in U-slots on a standardized rack. The positioning arm approaches the rack and engages the selected wrist through a mechanical interlock to ensure proper, and safe, seating. It then moves away from the rack by sliding the grooved flange of the wrist along the U-slot. Several considerations impact upon the implementation of the concept. The physical disengagement of the arm and the wrists requires the use of *direct-drive* wrist mechanisms actuated by *rare-earth* motors to achieve high torque-to-weight ratios [7, Appendix A]. Task geometry and constraints largely determine the configuration, type of connection and number of interchangeable wrists needed for a family of tasks. *A priori* knowledge of the presentation and handling requirements of the parts to be manipulated can facilitate the wrist design: the physical dimensions of the parts influence directly the size and articulation of the wrist, and proper fixturing (i.e., reliable part positioning and orientation) can reduce the complexity of the wrist [8]. The mechanical interlock that secures the wrist on the arm must also provide power and signal lines to the wrist. Microprocessors can then control the wrist and coordinate the wrist functions with the robot arm motion.

Although we believe that the interchangeable wrist concept can be a viable approach for increasing the flexibility of robot manipulators, it does not mean that it is applicable in all cases. In assembly operations, where positioning accuracy is critical, it may be more practical to use a multifunctional wrist (or a toolchanger) to reduce the number of times the robot arm must be positioned. Also, high-speed material handling,

Figure 1: *An Illustration of the Interchangeable Wrist Concept.*

which does not add value to a product, does not justify the cost of multiple wrists. We feel that interchangeable wrists are best suited for manufacturing processes such as machining, deburring, drilling, etc. A robot manipulator with interchangeable wrists can perform more of these operations on a fixtured part, and thus add more value to the part by eliminating secondary operations and reducing transport and set-up time. Interchangeable wrists can provide an attractive, practical alternative for manufacturing by covering the middle ground between low-cost, dedicated mechanisms and expensive, research-oriented, multifunctional designs. Furthermore, the use of interchangeable wrists may be the only alternative for teleoperators where human intervention is not possible (or desirable).

The objective of this paper is to determine the feasibility of the proposed concept for *real-time* applications. We present an analytical framework that incorporates the wrist-exchange procedure in the robot control hierarchy and demonstrate the applicability of our approach for the case of manipulators with interchangeable spherical wrists. We concentrate upon spherical wrists because of their superior orientational properties [9, p. 1300]. Our algorithm utilizes the individual kinematic properties of the arm and the wrists to generate an efficient solution to the kinematic control problem of the coupled system.

This paper is organized as follows. In Section 2, we highlight robot kinematics and introduce our notation. In Section 3, we describe the positioning arm and the orienting wrists and discuss specifications for wrist attachment. Then, in Section 4, we present our interchangeable wrist concept and, in Section 5, we develop a kinematic algorithm for the combined manipulator system. We illustrate our approach through the Unimation/Westinghouse PUMA 560 robot in Section 6. Finally, in Section 7, we summarize our results and highlight directions for future research.

2 Robot Kinematics

Robotic manipulators are articulated, open kinematic chains of N rigid bodies (links) which are connected serially by N joints. The links are numbered consecutively from the base (link 0) to the final end (link N). The joints are the points of articulation between the links and are numbered so that joint i connects links $(i-1)$ and i . To describe the geometry of robot motion, we assign a cartesian coordinate frame $C_i \equiv (O_{C_i}; \mathbf{x}_{C_i}, \mathbf{y}_{C_i}, \mathbf{z}_{C_i})$ to each link, where $i = 0, 1, 2, \dots, N$, according to the Denavit-Hartenberg convention [10, pp. 36–41]. The appeal of the Denavit-Hartenberg convention lies in its algorithmic approach. Each link translates along or rotates around the $\mathbf{z}$ -axis of the coordinate frame assigned to the previous link. The first link moves with respect to the $\mathbf{z}$ -axis of the zero (or base) coordinate frame $C_0 \equiv (O_{C_0}; \mathbf{x}_{C_0}, \mathbf{y}_{C_0}, \mathbf{z}_{C_0})$ which serves as a reference frame. The origin O_{C_0} of the zero frame C_0 can be placed anywhere in the supporting base and the position and orientation of the last coordinate frame C_N is specified by the geometry of the link N .

We proceed to model the kinematics of the robotic mechanism in terms of the joint coordinates. The joint coordinate q_i is defined as the angular displacement around $\mathbf{z}_{C_{i-1}}$, if joint i is rotational, or as the linear displacement along $\mathbf{z}_{C_{i-1}}$, if joint i is translational. The N -dimensional space defined by the joint coordinates $(q_1, \dots, q_N)$ is called the *configuration space* of the mechanism.

In the Denavit-Hartenberg convention, the relative position and orientation of two adjacent link coordinate frames can be described by a sequence of four elementary transformations (Figure 2):

- Rotation by an angle ϕ_{C_i} (in the right-handed sense) about the $\mathbf{z}_{C_{i-1}}$ axis, so that the $\mathbf{x}_{C_{i-1}}$ axis becomes parallel to the $\mathbf{x}_{C_i}$ axis.
- Translation by a distance of r_{C_i} along the positive direction of the $\mathbf{z}_{C_{i-1}}$ axis, to

Figure 2: The Denavit-Hartenberg Convention for Link Coordinate Frames.

align the $x_{C_{i-1}}$ axis with the x_{C_i} axis.

- Translation by a distance of d_{C_i} along the positive direction of the $x_{C_{i-1}} = x_{C_i}$ axis, to coalesce the origins $O_{C_{i-1}}$ and O_{C_i} .
- Rotation by an angle α_{C_i} (in the right-handed sense) about the $x_{C_{i-1}} = x_{C_i}$ axis, to coalesce the two coordinate frames.

Accordingly, the position vector of the origin O_{C_i} in the C_{i-1} frame is

$${}^{C_{i-1}}\mathbf{p}_{C_i} = \begin{pmatrix} d_{C_i} \cos \phi_{C_i} \\ d_{C_i} \sin \phi_{C_i} \\ r_{C_i} \end{pmatrix}, \quad (1)$$

while the orientation of frame C_i is represented by the orthogonal (3×3) rotation matrix

$$\begin{aligned} {}^{C_{i-1}}\mathbf{R}_{C_i} &= \begin{pmatrix} {}^{C_{i-1}}\mathbf{x}_{C_i} & {}^{C_{i-1}}\mathbf{y}_{C_i} & {}^{C_{i-1}}\mathbf{z}_{C_i} \end{pmatrix} \\ &= \begin{pmatrix} \cos \phi_{C_i} & -\cos \alpha_{C_i} \sin \phi_{C_i} & \sin \alpha_{C_i} \sin \phi_{C_i} \\ \sin \phi_{C_i} & \cos \alpha_{C_i} \cos \phi_{C_i} & -\sin \alpha_{C_i} \cos \phi_{C_i} \\ 0 & \sin \alpha_{C_i} & \cos \alpha_{C_i} \end{pmatrix}. \end{aligned} \quad (2)$$

Link i	ϕ_{C_i}	r_{C_i}	d_{C_i}	α_{C_i}	Range
1	q_1	0.0 mm	0.0 mm	-90.0°	$-160^\circ \leq q_1 \leq 160^\circ$
2	q_2	149.5 mm	432.0 mm	0.0°	$-225^\circ \leq q_2 \leq 45^\circ$
3	q_3	0.0 mm	0.0 mm	90.0°	$-45^\circ \leq q_3 \leq 225^\circ$
4	q_4	432.0 mm	0.0 mm	-90.0°	$-110^\circ \leq q_4 \leq 170^\circ$
5	q_5	0.0 mm	0.0 mm	90.0°	$-100^\circ \leq q_5 \leq 100^\circ$
6	q_6	56.5 mm	0.0 mm	0.0°	$-266^\circ \leq q_6 \leq 266^\circ$

Table 1: *Kinematic Parameters of the PUMA 560 Robot.*

In (1) and (2), r_{C_i} and ϕ_{C_i} are the distance and angle between links $(i-1)$ and i ; d_{C_i} and α_{C_i} are the length and twist of link i . If joint i is rotational, then $q_i = \phi_{C_i}$, and α_{C_i} , d_{C_i} , and r_{C_i} are constant geometric parameters; if joint i is translational, then $q_i = r_{C_i}$, and d_{C_i} , α_{C_i} and ϕ_{C_i} are constant geometric parameters. The leading superscript signifies the reference frame in which the corresponding vector or matrix is expressed.

In Figure 3, we illustrate the definition of the link coordinate frames for a popular industrial manipulator, the PUMA 560. (The associated kinematic parameters are summarized in Table 1.) The base frame C_0 serves as the global reference frame. The coordinate axes of the end-effector frame $C_N = C_6$ are commonly referred to as the normal (n), sliding (s), and approach (a) unit vectors respectively. A manipulator task is defined by the time-evolution of the desired position and orientation of the end-effector frame:

$$\begin{aligned}
 \text{desired position} \quad {}^{C_0}\mathbf{p}_{C_N}(t) &= \begin{pmatrix} {}^{C_0}p_x \\ {}^{C_0}p_y \\ {}^{C_0}p_z \end{pmatrix} \\
 \text{desired orientation} \quad {}^{C_0}\mathbf{R}_{C_N}(t) &= \begin{pmatrix} {}^{C_0}n_x & {}^{C_0}s_x & {}^{C_0}a_x \\ {}^{C_0}n_y & {}^{C_0}s_y & {}^{C_0}a_y \\ {}^{C_0}n_z & {}^{C_0}s_z & {}^{C_0}a_z \end{pmatrix}
 \end{aligned} \tag{3}$$

Throughout this paper, we use *geometrical* (3×1) vectors, rather than the (4×1) vectors of the homogeneous transformation formalism [10]. The multiplication of a geometrical vector by a rotation matrix preserves the *magnitude* of the vector; i.e.,

${}^{C_i}R_{C_{i+1}} {}^{C_{i+1}}p_{C_{i+2}} \neq {}^{C_i}p_{C_{i+2}}$ in Figure 4. Rather,

$${}^{C_i}p_{C_{i+2}} = {}^{C_i}p_{C_{i+1}} + {}^{C_i}R_{C_{i+1}} {}^{C_{i+1}}p_{C_{i+2}}. \quad (4)$$

Accordingly, the motion of the end-effector is computed as follows:

$${}^{C_0}p_{C_N} = {}^{C_0}p_{C_1} + {}^{C_0}R_{C_1} {}^{C_1}p_{C_2} + \dots + {}^{C_0}R_{C_1} {}^{C_1}R_{C_2} \dots {}^{C_{N-2}}R_{C_{N-1}} {}^{C_{N-1}}p_{C_N}. \quad (5)$$

$${}^{C_0}R_{C_N} = {}^{C_0}R_{C_1} {}^{C_1}R_{C_2} \dots {}^{C_{N-1}}R_{C_N}. \quad (6)$$

In robotics, the *inverse kinematics* problem deals with the computation of the joint coordinate profiles $q_1(t), q_2(t), \dots, q_N(t)$ that are necessary to produce the desired motion of the end-effector in (3). Inverse kinematic algorithms are computationally complex and depend upon the manipulator geometry. Closed-form solutions to the inverse kinematics problem do not always exist and one has to resort to numerical analysis techniques to solve for the joint coordinates [11,12]. Analytical solutions, however, exist for low-order mechanisms [13, pp. 103–104]; these solutions are customized for specific manipulator geometries.

3 Arm and Wrist Coordinate Frames

Robotic manipulators consist of two distinct subsystems: the positioning arm and the orienting wrist. In an N -joint robot, the first M ($M < N$) joints constitute the arm and the remaining $(N - M)$ joints constitute the wrist. The arm is responsible for positioning the wrist at a specified point in the workspace while the wrist is responsible for orienting the end-effector. The arm may consist of rotational and translational joints while the wrist typically has all rotational joints. For a robot to be able to position and orient its end-effector in a three-dimensional workspace, the robot must have at least six joints, providing six degrees-of-freedom. Typical industrial manipulators have six joints that imitate the human hand: the first three joints constitute the arm and the remaining three constitute the wrist.

The arm and wrist are two subsystems with different tasks. Closed-form solutions to the inverse kinematics problem for each subsystem exist since the arm and the wrist are low-order mechanisms. Connecting the wrist to the arm, however, creates a highly coupled system. Although the arm is primarily responsible for positioning, motion of any of its joints can destroy the orientation of the wrist through rotational side-effects. Similarly, motion of the joints of the wrist to achieve a desired orientation can have translational side-effects on the end-effector position. The applicability of our interchangeable wrist concept depends upon the real-time coordination of the arm and the wrists. In Section 4, we prove that the inverse kinematics solutions for the arm and a wrist can be utilized to solve the inverse kinematics problem of the manipulator, without having to solve explicitly for the combined system. The solution involves adjusting the joint coordinates for each subsystem and taking into consideration the translational and rotational side-effects of motions of accommodation. Our approach depends upon a new set of coordinate frames that are independently assigned to the arm and the wrists. We detail these coordinate frame assignments in turn. For expository convenience, and without losing generality, we consider six degree-of-freedom manipulators with the

positioning and orienting subsystems having three joints each.

3.1 The Positioning Arm

The positioning arm is a three degree-of-freedom mechanism that consists of three links. The orienting wrist is physically attached to the last link of the arm. The robot arm is responsible for positioning the wrist attachment point.

To describe arm motion, we assign a cartesian coordinate frame $A_i \equiv (O_{A_i}; \mathbf{x}_{A_i}, \mathbf{y}_{A_i}, \mathbf{z}_{A_i})$ to each link ($i = 0, 1, 2, 3$) according to the Denavit-Hartenberg convention. The zero frame A_0 serves as the reference frame while the third frame A_3 is fixed at the wrist attachment point. The Denavit-Hartenberg convention allows some freedom of choice regarding the orientation of frame A_3 . To simplify the kinematic computations of the arm, we specify $\mathbf{z}_{A_3} \parallel \mathbf{z}_{A_2}$ so that the twist of the third link is zero, $\alpha_{A_3} \equiv 0^\circ$. The configuration space of the arm is defined by the joint coordinates $(\theta_{A_1}, \theta_{A_2}, \theta_{A_3})$.

We point out that the arm coordinate frames are not necessarily the same frames that one would choose by examining the combined manipulator. For instance, $\mathbf{z}_{A_3}$ may *not* be the axis of rotation of the first wrist joint; the robot arm is designed to accommodate a variety of wrist types and there is no *a priori* knowledge of the wrist choice. The choice of arm coordinate frames may also add a fixed offset to the zero reference of the joint coordinates that characterize the entire manipulator.

In Figure 3, we display our choice of arm coordinate frames for the positioning system of the PUMA 560 robot along with the conventional robot frames. (The associated kinematic parameters are summarized in Table 2.) We observe that:

$$A_0 \equiv C_0 \quad ; \quad A_1 \equiv C_1 \quad ; \quad A_2 \equiv C_2 \quad ; \quad A_3 \neq C_3$$

$$q_1 \equiv \theta_{A_1} \quad ; \quad q_2 \equiv \theta_{A_2} \quad ; \quad q_3 \equiv \theta_{A_3} + 90^\circ.$$

The inverse kinematics problem of the arm deals with the computation of the arm joint

Link i	ϕ_{A_i}	r_{A_i}	d_{A_i}	α_{A_i}	Range
1	θ_{A_1}	0.0 mm	0.0 mm	-90.0°	$-160^\circ \leq \theta_{A_1} \leq 160^\circ$
2	θ_{A_2}	149.5 mm	432.0 mm	0.0°	$-225^\circ \leq \theta_{A_2} \leq 45^\circ$
3	θ_{A_3}	0.0 mm	432.0 mm	0.0°	$-135^\circ \leq \theta_{A_3} \leq 135^\circ$

Table 2: Kinematic Parameters of the PUMA 560 Arm.

coordinates $\theta_A = (\theta_{A_1}, \theta_{A_2}, \theta_{A_3})^T$ that are necessary to produce the desired position of the wrist attachment point ${}^{A_0}\mathbf{p}_{A_3}$:

$$\begin{aligned} {}^{A_0}\mathbf{p}_{A_3} &= {}^{A_0}\mathbf{p}_{A_1} + {}^{A_0}\mathbf{R}_{A_1} {}^{A_1}\mathbf{p}_{A_2} + {}^{A_0}\mathbf{R}_{A_1} {}^{A_1}\mathbf{R}_{A_2} {}^{A_2}\mathbf{p}_{A_3} = \mathbf{f}_t(\theta_A) \iff \\ &\iff \theta_A = \mathbf{f}_t^{-1}({}^{A_0}\mathbf{p}_{A_3}). \end{aligned} \quad (7)$$

We assume, without loss of generality, that there is a closed-form solution to this inverse problem; i.e., an analytic form of the vector function $\mathbf{f}_t^{-1}(\cdot)$ in (7) exists.

The choice of arm joint coordinates in (7) implicitly determines the orientation of frame A_3 :

$${}^{A_0}\mathbf{R}_{A_3} = {}^{A_0}\mathbf{R}_{A_1} {}^{A_1}\mathbf{R}_{A_2} {}^{A_2}\mathbf{R}_{A_3} = \mathbf{f}_r(\theta_A). \quad (8)$$

In (7) and (8), the position vectors and orientation matrices of adjacent frames are defined as in (1) and (2), respectively. We point out that the functions $\mathbf{f}_t(\cdot)$, $\mathbf{f}_t^{-1}(\cdot)$ and $\mathbf{f}_r(\cdot)$ in (7) and (8) are functions of the *elements* of the arguments, rather than of the vectors θ_A and ${}^{A_0}\mathbf{p}_{A_3}$ themselves. We have adopted this symbolic convention for expository convenience.

3.2 The Orienting Wrist

The orienting wrist is a three degree-of-freedom mechanism that consists of three rotational links. The orienting wrist is responsible for orienting the end-effector.

To describe wrist motion, we assign a cartesian coordinate frame $W_i \equiv (O_{W_i}; \mathbf{x}_{W_i}, \mathbf{y}_{W_i}, \mathbf{z}_{W_i})$ to each link ($i = 0, 1, 2, 3$) according to the Denavit-Hartenberg convention. The

Link i	ϕ_{W_i}	r_{W_i}	d_{W_i}	α_{W_i}	Range
1	θ_{W_1}	0.0 mm	0.0 mm	-90.0°	$-110^\circ \leq \theta_{W_1} \leq 170^\circ$
2	θ_{W_2}	0.0 mm	0.0 mm	90.0°	$-100^\circ \leq \theta_{W_2} \leq 100^\circ$
3	θ_{W_3}	56.5 mm	0.0 mm	0.0°	$-266^\circ \leq \theta_{W_3} \leq 266^\circ$

Table 3: *Kinematic Parameters of the PUMA 560 Wrist.*

zero frame W_0 serves as the reference frame while the third frame W_3 is the end-effector (or task) frame. The configuration space of the wrist is specified by the wrist joint coordinates $(\theta_{W_1}, \theta_{W_2}, \theta_{W_3})$.

We note that the wrist coordinate frames are not necessarily the frames that one would choose by examining the combined manipulator. The choice of wrist coordinate frames may result in a fixed offset of the zero reference of the joint coordinates that characterize the entire manipulator.

In Figure 3, we display our choice of wrist coordinate frames for the orienting system of the PUMA 560 robot along with the conventional robot frames. (The associated kinematic parameters are summarized in Table 3.) We observe that:

$$W_0 \neq C_3 \quad ; \quad W_1 \equiv C_4 \quad ; \quad W_2 \equiv C_5 \quad ; \quad W_3 \equiv C_6$$

$$q_4 \equiv \theta_{W_1} \quad ; \quad q_5 \equiv \theta_{W_2} \quad ; \quad q_6 \equiv \theta_{W_3}.$$

The inverse kinematics problem of the wrist deals with the computation of the wrist joint coordinates $\theta_W = (\theta_{W_1}, \theta_{W_2}, \theta_{W_3})^T$ that are necessary to produce the desired orientation of the end-effector ${}^{W_0}\mathbf{R}_{W_3}$:

$${}^{W_0}\mathbf{R}_{W_3} = {}^{W_0}\mathbf{R}_{W_1} {}^{W_1}\mathbf{R}_{W_2} {}^{W_2}\mathbf{R}_{W_3} = \mathbf{g}_r(\theta_W) \iff \theta_W = \mathbf{g}_r^{-1}({}^{W_0}\mathbf{R}_{W_3}). \quad (9)$$

We assume, without loss of generality, that there is a closed-form solution to this inverse problem; i.e., an analytic form of the vector function $\mathbf{g}_r^{-1}(\cdot)$ in (9) exists.

The choice of wrist joint coordinates in (9) implicitly determines the position of the origin of frame W_3 :

$${}^{W_0}\mathbf{p}_{W_3} = {}^{W_0}\mathbf{p}_{W_1} + {}^{W_0}\mathbf{R}_{W_1} {}^{W_1}\mathbf{p}_{W_2} + {}^{W_0}\mathbf{R}_{W_1} {}^{W_1}\mathbf{R}_{W_2} {}^{W_2}\mathbf{p}_{W_3} = \mathbf{g}_t(\boldsymbol{\theta}_W). \quad (10)$$

In (9) and (10), the position vectors and orientation matrices of adjacent frames are defined as in (1) and (2), respectively. As in the previous section, the functions $\mathbf{g}_r(\cdot)$, $\mathbf{g}_r^{-1}(\cdot)$ and $\mathbf{g}_t(\cdot)$ in (9) and (10) are functions of the *elements* of the arguments, rather than of the vector $\boldsymbol{\theta}_W$ and matrix ${}^{W_0}\mathbf{R}_{W_3}$ themselves.

3.3 Arm-Wrist Attachment

When the positioning arm engages a selected wrist, the mechanical interlock guarantees proper and safe seating. The arm-wrist attachment is such that $O_{A_3} \equiv O_{W_0}$. The relative arm-wrist orientation is described by a *fixed* (time-independent) rotation matrix ${}^{A_3}\mathbf{R}_{W_0}$. This rotation matrix completely characterizes the arm-wrist attachment and depends explicitly upon the individual wrist geometry. Typically, a wrist is attached to an arm with its first axis of rotation parallel (or perpendicular) to the axial direction of the last link of the arm [14].

4 The Interchangeable Wrist Concept

The successful completion of a robotic task depends upon the coordination of the motions of the arm and wrist. Although the arm is primarily responsible for positioning and the wrist is responsible for orienting the end-effector, there are *coupling* side-effects. Positioning the wrist attachment point at a desired location results in an implicit orientation of the wrist:

$$\text{Primary Arm Function} \quad {}^{A_0}\mathbf{p}_{A_3} = \mathbf{f}_t(\boldsymbol{\theta}_A) \quad (11)$$

$$\text{Secondary Arm Function} \quad {}^{A_0}\mathbf{R}_{A_3} = \mathbf{f}_r(\boldsymbol{\theta}_A) \quad (12)$$

Similarly, orienting motions of the wrist result in secondary positioning of the end-effector:

$$\text{Primary Wrist Function} \quad {}^{W_0}\mathbf{R}_{W_3} = \mathbf{g}_r(\theta_W) \quad (13)$$

$$\text{Secondary Wrist Function} \quad {}^{W_0}\mathbf{p}_{W_3} = \mathbf{g}_t(\theta_W) \quad (14)$$

The desired position and orientation of the end-effector are typically expressed in a stationary frame such as the arm reference frame:

$$\begin{aligned} {}^{A_0}\mathbf{p}_E(t) &= \begin{pmatrix} {}^{A_0}p_{E_x} \\ {}^{A_0}p_{E_y} \\ {}^{A_0}p_{E_z} \end{pmatrix} \\ {}^{A_0}\mathbf{R}_E(t) &= \begin{pmatrix} {}^{A_0}n_x & {}^{A_0}s_x & {}^{A_0}a_x \\ {}^{A_0}n_y & {}^{A_0}s_y & {}^{A_0}a_y \\ {}^{A_0}n_z & {}^{A_0}s_z & {}^{A_0}a_z \end{pmatrix}. \end{aligned} \quad (15)$$

The feasibility of the proposed interchangeable wrist concept depends upon the ability to define the primary arm and wrist responsibilities, ${}^{A_0}\mathbf{p}_{A_3}$ and ${}^{W_0}\mathbf{R}_{W_3}$, from the task description in (15), and to account for the secondary motions, ${}^{A_0}\mathbf{R}_{A_3}$ and ${}^{W_0}\mathbf{p}_{W_3}$. In the sequence, we formulate analytically these objectives. The development is based upon the schematic diagram in Figure 5.

4.1 Task Decomposition

The wrist attachment guarantees that $O_{A_3} \equiv O_{W_0}$. Hence:

$${}^{A_0}\mathbf{p}_{A_3} = {}^{A_0}\mathbf{p}_{W_0} = {}^{A_0}\mathbf{p}_E - {}^{A_0}\mathbf{R}_{W_0} {}^{W_0}\mathbf{p}_{W_3}. \quad (16)$$

Substituting (9) and (14) and into (16), we have:

$${}^{A_0}\mathbf{p}_{A_3} = {}^{A_0}\mathbf{p}_E - {}^{A_0}\mathbf{R}_{W_0} \mathbf{g}_t(\theta_W) = {}^{A_0}\mathbf{p}_E - {}^{A_0}\mathbf{R}_{W_0} \mathbf{g}_t[\mathbf{g}_r^{-1}({}^{W_0}\mathbf{R}_{W_3})]. \quad (17)$$

We observe that the primary tasks of the arm and wrist, ${}^{A_0}\mathbf{p}_{A_3}$ and ${}^{W_0}\mathbf{R}_{W_3}$ respectively, are intimately related through (17). Decoupling the two tasks is possible

Figure 5: *Schematic Representation of Arm and Wrist Coordinate Frames.*

if and only if the rotation matrix ${}^{A_0}\mathbf{R}_{W_0}$ and the vector function $\mathbf{g}_t[\mathbf{g}_r^{-1}(\cdot)]$ satisfy the associative property for multiplication:

$${}^{A_0}\mathbf{R}_{W_0} \mathbf{g}_t[\mathbf{g}_r^{-1}({}^{W_0}\mathbf{R}_{W_3})] = \mathbf{g}_t[\mathbf{g}_r^{-1}({}^{A_0}\mathbf{R}_{W_0} {}^{W_0}\mathbf{R}_{W_3})] = \mathbf{g}_t[\mathbf{g}_r^{-1}({}^{A_0}\mathbf{R}_{W_3})]. \quad (18)$$

We note that the condition in (18) depends explicitly upon the wrist functions $\mathbf{g}_t(\cdot)$ and $\mathbf{g}_r^{-1}(\cdot)$. If the wrist geometry is such that (18) is satisfied, the positioning task of the arm can be readily evaluated from the task description in (15):

$${}^{A_0}\mathbf{p}_{A_3} = {}^{A_0}\mathbf{p}_E - \mathbf{g}_t[\mathbf{g}_r^{-1}({}^{A_0}\mathbf{R}_{W_3})] = {}^{A_0}\mathbf{p}_E - \mathbf{g}_t[\mathbf{g}_r^{-1}({}^{A_0}\mathbf{R}_E)]. \quad (19)$$

The desired arm coordinates are then computed from (7). Once the position of the wrist attachment has been fixed, the rotational side-effect of positioning the arm, ${}^{A_0}\mathbf{R}_{A_3}$, can be computed from (12). The orienting task of the wrist is defined as follows:

$$\begin{aligned} {}^{A_0}\mathbf{R}_{A_3} {}^{A_3}\mathbf{R}_{W_0} {}^{W_0}\mathbf{R}_{W_3} &= {}^{A_0}\mathbf{R}_{W_3} \iff {}^{W_0}\mathbf{R}_{W_3} = ({}^{A_3}\mathbf{R}_{W_0})^{-1} ({}^{A_0}\mathbf{R}_{A_3})^{-1} {}^{A_0}\mathbf{R}_{W_3} \\ &\iff {}^{W_0}\mathbf{R}_{W_3} = ({}^{A_3}\mathbf{R}_{W_0})^T [\mathbf{f}_r(\theta_A)]^T {}^{A_0}\mathbf{R}_E. \end{aligned} \quad (20)$$

The desired wrist coordinates are then computed from (9), and the development is complete since the translational side-effects of the orienting wrist have been accounted for in (17).

4.2 Spherical Wrists

In Appendix A, we prove analytically that the condition in (18) is satisfied for *spherical* wrists. Spherical wrists consist of three rotational joints with the axes of rotation intersecting at a common point. Spherical wrists permit continuous tool rotation and simplify the design of the end-effector. Moreover, the spherical wrist geometry guarantees a decoupling of the orientation of the end-effector from the position of the manipulator [13, pp. 103–104].

The proof in Appendix A, which is a contribution of our research, demonstrates that the spherical wrist geometry is the *only* wrist geometry that satisfies condition (18). This result does not reduce the utility of the interchangeable wrist concept, since the majority of current industrial manipulators employ *spherical* wrists [15, p. 183]. (The wrist of the PUMA 560 robot in Figure 3 is an example of spherical wrist geometry.)

For non-spherical wrist geometries, condition (18) is not satisfied and the primary tasks of the arm and wrist remain coupled in (17). The modification of our algorithm to accommodate non-spherical wrists is a subject of current research.

5 The Inverse Kinematics Algorithm

In this section, we summarize our inverse kinematics algorithm. A block diagram of the proposed algorithm appears in Figure 6. The shaded blocks signify closed-form, analytic solutions that have been developed and tested off-line for the arm and individual wrists. Each wrist configuration is codified and the pertinent geometric

<u>STEP 1</u>	Read desired position and orientation of the end-effector: ${}^{A_0}\mathbf{p}_E, {}^{A_0}\mathbf{R}_E$.
<u>STEP 2</u>	Compute desired position of wrist attachment: ${}^{A_0}\mathbf{p}_{A_3} = {}^{A_0}\mathbf{p}_E - \mathbf{g}_t[\mathbf{g}_r^{-1}({}^{A_0}\mathbf{R}_E)]$.
<u>STEP 3</u>	Compute desired arm joint coordinates: $\boldsymbol{\theta}_A = \mathbf{f}_t^{-1}({}^{A_0}\mathbf{p}_{A_3})$.
<u>STEP 4</u>	Read wrist attachment specification: ${}^{A_3}\mathbf{R}_{W_0}$.
<u>STEP 5</u>	Compute desired wrist orientation: ${}^{W_0}\mathbf{R}_{W_3} = ({}^{A_3}\mathbf{R}_{W_0})^T[\mathbf{f}_r(\boldsymbol{\theta}_A)]^T {}^{A_0}\mathbf{R}_E$.
<u>STEP 6</u>	Compute desired wrist joint coordinates: $\boldsymbol{\theta}_W = \mathbf{g}_r^{-1}({}^{W_0}\mathbf{R}_{W_3})$.

Table 4: *Inverse Kinematics Algorithm for Manipulators with Interchangeable Spherical Wrists.*

parameters, the wrist solution, and the arm/wrist connection are recorded permanently in the database of the controller. Once the robot arm engages a particular wrist, the wrist code is identified and the corresponding algorithmic blocks are employed in the inverse kinematics solution. The proposed framework is *general-purpose* and implementable in *real-time*. The computational steps of the proposed inverse kinematics algorithm for manipulators with interchangeable *spherical* wrists are enumerated in Table 4.

6 An Example: The PUMA 560 Robot

In this section, we demonstrate the accuracy and applicability of our algorithm through the PUMA 560 robot. This six degree-of-freedom, computer-controlled industrial manipulator has been frequently used to test and compare the efficiency of kinematic control algorithms [16].

6.1 The PUMA 560 Arm

The inverse kinematics solution $\theta_A = \mathbf{f}_t^{-1}({}^{A_0}\mathbf{p}_{A_3})$ for the PUMA 560 arm (Figure 3) is obtained from [10, pp. 60–76] with minor adjustments to account for the modification $A_3 \neq C_3$:

$$\begin{aligned}\theta_{A_1} &= \text{ATAN2} \left(\frac{-{}^{A_0}p_{A_3x}r_{A_2} - \text{ARM } {}^{A_0}p_{A_3y}\sqrt{{}^{A_0}p_{A_3x}^2 + {}^{A_0}p_{A_3y}^2 - r_{A_2}^2}}{{}^{A_0}p_{A_3y}r_{A_2} - \text{ARM } {}^{A_0}p_{A_3x}\sqrt{{}^{A_0}p_{A_3x}^2 + {}^{A_0}p_{A_3y}^2 - r_{A_2}^2}} \right) \\ \theta_{A_3} &= \text{ATAN2} \left(\frac{\text{ELBOW } \sqrt{4d_{A_2}^2d_{A_3}^2 - ({}^{A_0}p_{A_3x}^2 + {}^{A_0}p_{A_3y}^2 + {}^{A_0}p_{A_3z}^2 - d_{A_3}^2 - d_{A_2}^2 - r_{A_2}^2)^2}}{{}^{A_0}p_{A_3x}^2 + {}^{A_0}p_{A_3y}^2 + {}^{A_0}p_{A_3z}^2 - d_{A_3}^2 - d_{A_2}^2 - r_{A_2}^2}} \right) \\ \theta_{A_2} &= \text{ATAN2} \left(\frac{-d_{A_3} \sin \theta_{A_3} ({}^{A_0}p_{A_3x} \cos \theta_{A_1} + {}^{A_0}p_{A_3y} \sin \theta_{A_1}) - {}^{A_0}p_{A_3z} (d_{A_3} \cos \theta_{A_3} + d_{A_2})}{(d_{A_3} \cos \theta_{A_3} + d_{A_2}) ({}^{A_0}p_{A_3x} \cos \theta_{A_1} + {}^{A_0}p_{A_3y} \sin \theta_{A_1}) - {}^{A_0}p_{A_3z} d_{A_3} \sin \theta_{A_3}} \right).\end{aligned}$$

The binary indicators $\text{ARM} = \pm 1$ and $\text{ELBOW} = \pm 1$ signify one of the four possible solutions [10, p. 63].

The secondary arm function ${}^{A_0}\mathbf{R}_{A_3} = \mathbf{f}_r(\theta_A)$ is:

$${}^{A_0}\mathbf{R}_{A_3} = \begin{pmatrix} \cos \theta_{A_1} \cos(\theta_{A_2} + \theta_{A_3}) & -\cos \theta_{A_1} \sin(\theta_{A_2} + \theta_{A_3}) & -\sin \theta_{A_1} \\ \sin \theta_{A_1} \cos(\theta_{A_2} + \theta_{A_3}) & -\sin \theta_{A_1} \sin(\theta_{A_2} + \theta_{A_3}) & \cos \theta_{A_1} \\ -\sin(\theta_{A_2} + \theta_{A_3}) & -\cos(\theta_{A_2} + \theta_{A_3}) & 0 \end{pmatrix}.$$

6.2 The PUMA 560 Wrist

The PUMA 560 employs a spherical wrist whose joint coordinates define the orientation of the end-effector in Euler angles. The inverse kinematics solution $\theta_W = \mathbf{g}_r^{-1}({}^{W_0}\mathbf{R}_{W_3})$ for the PUMA 560 wrist (Figure 3) is:

$$\begin{aligned}\theta_{W_1} &= \text{ATAN2} \left(\frac{\text{WRIST } {}^{W_0}a_{W_3y}}{\text{WRIST } {}^{W_0}a_{W_3x}} \right) \\ \theta_{W_2} &= \text{ATAN2} \left(\frac{\text{WRIST } \sqrt{1 - {}^{W_0}a_{W_3x}^2}}{{}^{W_0}a_{W_3x}} \right) \\ \theta_{W_3} &= \text{ATAN2} \left(\frac{\text{WRIST } {}^{W_0}s_{W_3x}}{-\text{WRIST } {}^{W_0}n_{W_3x}} \right)\end{aligned}$$

where the binary indicator $\text{WRIST} = \pm 1$ identifies one of the two possible solutions [17] and $|{}^{W_0}a_{W_3x}| + |{}^{W_0}a_{W_3y}| \neq 0$. If ${}^{W_0}a_{W_3x} = {}^{W_0}a_{W_3y} = 0$ (degenerate case), θ_{W_1} and θ_{W_3} describe the same orientation and cannot be computed separately [17]. If the degenerate case occurs, a practical strategy is to restrict the motion of the major joint and allow θ_{W_1} to retain its current value:

$$\begin{aligned}\theta_{W_1} &= \text{current value} \\ \theta_{W_2} &= 0^\circ \\ \theta_{W_3} &= -\theta_{W_1} + \text{ATAN2}\left(\frac{{}^{W_0}n_{W_3y}}{{}^{W_0}n_{W_3x}}\right).\end{aligned}$$

The secondary wrist function ${}^{W_0}\mathbf{p}_{W_3} = \mathbf{g}_t(\theta_W)$ is:

$${}^{W_0}\mathbf{p}_{W_3} = \begin{pmatrix} r_{W_3} \cos \theta_{W_1} \sin \theta_{W_2} \\ r_{W_3} \sin \theta_{W_1} \sin \theta_{W_2} \\ r_{W_3} \cos \theta_{W_2} \end{pmatrix}.$$

The wrist attachment of the PUMA 560 robot is such that the first axis of rotation of the wrist is parallel to the axial direction of the last link of the arm, i.e. $\mathbf{z}_{W_0} \parallel \mathbf{x}_{A_3}$, as shown in Figure 3. The rotation matrix that relates frames W_0 and A_3 is therefore:

$${}^{A_3}\mathbf{R}_{W_0} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

6.3 Numerical Example

Our numerical example involves solving for the joint coordinates needed to achieve a given position and orientation of the end-effector. For expository convenience, we assume that the manipulator approaches the desired position and orientation from a current configuration that corresponds to $\text{ARM} = -1$, $\text{ELBOW} = -1$, and $\text{WRIST} = 1$. (This is the PUMA 560 configuration depicted in Figure 3.) We solve for the joint coordinates of the arm and wrist by following the steps enumerated in Table 4.

$$\underline{STEP 1} \quad {}^{A_0}\mathbf{p}_E = \begin{pmatrix} 756.0663 \\ -240.8214 \\ -200.2913 \end{pmatrix} ; \quad {}^{A_0}\mathbf{R}_E = \begin{pmatrix} 0.1815 & 0.0757 & 0.9805 \\ 0.9559 & -0.2478 & -0.1578 \\ 0.2310 & 0.9658 & -0.1174 \end{pmatrix}.$$

$$\underline{STEP 2} \quad {}^{A_0}\mathbf{p}_{A_3} = {}^{A_0}\mathbf{p}_E - \mathbf{g}_t[\mathbf{g}_r^{-1}({}^{A_0}\mathbf{R}_E)] = (700.6699, -231.9043, -193.6603)^T.$$

$$\underline{STEP 3} \quad \theta_A = \mathbf{f}_t^{-1}({}^{A_0}\mathbf{p}_{A_3}) = (-30.0000^\circ, 45.0000^\circ, -60.0000^\circ).$$

$$\underline{STEP 4} \quad {}^{A_3}\mathbf{R}_{W_0} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

$$\underline{STEP 5} \quad {}^{W_0}\mathbf{R}_{W_3} = ({}^{A_3}\mathbf{R}_{W_0})^T [\mathbf{f}_r(\theta_A)]^T {}^{A_0}\mathbf{R}_E = \begin{pmatrix} -0.3062 & -0.8839 & 0.3536 \\ 0.9186 & -0.1768 & 0.3536 \\ -0.2500 & 0.4330 & 0.8660 \end{pmatrix}.$$

$$\underline{STEP 6} \quad \theta_W = \mathbf{g}_r^{-1}({}^{W_0}\mathbf{R}_{W_3}) = (45.0000^\circ, 30.0000^\circ, 60.0000^\circ).$$

The computational cost of our algorithm is $[26a + 39m + 15f]$ where a , m , and f stand for additions, multiplications, and transcendental function calls. For comparison, we note that the most efficient *customized* inverse kinematics algorithm for the PUMA 560 robot [16] requires $[23a + 31m + 16f]$. Our algorithm thus requires approximately 20 % more operations; we believe that this is a reasonable penalty for a *general-purpose* algorithm that can be used with multiple spherical wrists. (In reality, a fraction of the difference in computational cost is due to the fact that the algorithm in [16] assumes zero tool length, $r_{W_3} = 0$, while our algorithm accommodates variable tool lengths.) Our numerical example, therefore, demonstrates the accuracy and computational efficiency of our approach.

7 Conclusions

In this paper, we introduced the concept of *interchangeable* wrists as a cost-effective, practical method of improving the flexibility of a robot system without increasing the complexity of the wrist mechanism. We demonstrated the feasibility of our concept by presenting an analytical framework for the kinematics of manipulators with interchangeable spherical wrists. We utilized the individual closed-form kinematic solutions of the arm and the wrists to generate a general-purpose algorithmic solution for the kinematic control problem of the coupled system. We then highlighted the accuracy and computational efficiency of our algorithm through a numerical example for the PUMA 560 robot.

Current research efforts are focusing upon:

- customization of our algorithm (in Table 4) for specific robot arm geometries to further reduce the computational cost; and
- modification of the block diagram (in Figure 6) to accommodate non-spherical wrist geometries.

Appendix A

The objective of this appendix is to prove that the spherical wrist geometry is the *only* wrist geometry that satisfies the associative property for multiplication:

$${}^{A_0}\mathbf{R}_{W_0}\mathbf{g}_t[\mathbf{g}_r^{-1}({}^{W_0}\mathbf{R}_{W_3})] = \mathbf{g}_t[\mathbf{g}_r^{-1}({}^{A_0}\mathbf{R}_{W_0} {}^{W_0}\mathbf{R}_{W_3})] = \mathbf{g}_t[\mathbf{g}_r^{-1}({}^{A_0}\mathbf{R}_{W_3})]. \quad (\text{A-1})$$

The linear transformation in (A-1) requires that the time-varying orthonormal triple $({}^{W_0}\mathbf{x}_{W_3}, {}^{W_0}\mathbf{y}_{W_3}, {}^{W_0}\mathbf{z}_{W_3})$ *spans* the vector space $\mathbf{g}_t[\mathbf{g}_r^{-1}(\cdot)]$ [18, Appendix A]. It suffices, therefore, to prove that the spherical wrist geometry is the only geometry for which:

$$\mathbf{g}_t[\mathbf{g}_r^{-1}({}^{W_0}\mathbf{R}_{W_3})] = k_1 {}^{W_0}\mathbf{x}_{W_3} + k_2 {}^{W_0}\mathbf{y}_{W_3} + k_3 {}^{W_0}\mathbf{z}_{W_3} \quad (\text{A-2})$$

where k_1 , k_2 , and k_3 are *constant* (time-independent) coefficients.

The vector function $\mathbf{g}_t[\mathbf{g}_r^{-1}({}^{W_0}\mathbf{R}_{W_3})]$ defines the position vector ${}^{W_0}\mathbf{p}_{W_3}$ for a given end-effector orientation ${}^{W_0}\mathbf{R}_{W_3}$. Utilizing the forward recursive equations that relate link coordinate frames [19], we can write:

$$\begin{aligned} {}^{W_0}\mathbf{p}_{W_3} &= \mathbf{g}_t[\mathbf{g}_r^{-1}({}^{W_0}\mathbf{R}_{W_3})] \\ &= r_{W_1} {}^{W_0}\mathbf{z}_{W_0} + d_{W_1} {}^{W_0}\mathbf{x}_{W_1} + r_{W_2} {}^{W_0}\mathbf{z}_{W_1} + d_{W_2} {}^{W_0}\mathbf{x}_{W_2} + r_{W_3} {}^{W_0}\mathbf{z}_{W_2} + d_{W_3} {}^{W_0}\mathbf{x}_{W_3}. \end{aligned} \quad (\text{A-3})$$

Our objective is to identify conditions for which the vectors in (A-3) can be expressed in terms of the time-varying *basis* $({}^{W_0}\mathbf{x}_{W_3}, {}^{W_0}\mathbf{y}_{W_3}, {}^{W_0}\mathbf{z}_{W_3})$. We examine each of these vectors in turn.

The first term in (A-3) is a constant vector,

$$r_{W_1} {}^{W_0}\mathbf{z}_{W_0} = r_{W_1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (\text{A-4})$$

while the second and third vector terms in (A-3),

$$d_{W_1} {}^{W_0}\mathbf{x}_{W_1} = d_{W_1} \begin{pmatrix} \cos \theta_{W_1} \\ \sin \theta_{W_1} \\ 0 \end{pmatrix} \quad \text{and} \quad r_{W_2} {}^{W_0}\mathbf{z}_{W_1} = r_{W_2} \begin{pmatrix} \sin \alpha_{W_1} \sin \theta_{W_1} \\ -\sin \alpha_{W_1} \cos \theta_{W_1} \\ \cos \alpha_{W_1} \end{pmatrix} \quad (\text{A-5})$$

respectively, have a constant third element. None of these vectors, therefore, can be expressed in terms of the time-varying basis $({}^{W_0}\mathbf{x}_{W_3}, {}^{W_0}\mathbf{y}_{W_3}, {}^{W_0}\mathbf{z}_{W_3})$. Accordingly, the condition in (A-2) will be satisfied if and only if the wrist geometry is such that $r_{W_1} = d_{W_1} = r_{W_2} = 0$.

The fourth vector term in (A-3) can be expanded as [19]:

$$d_{W_2} {}^{W_0}\mathbf{x}_{W_2} = (d_{W_2} \cos \theta_{W_3}) {}^{W_0}\mathbf{x}_{W_3} + (d_{W_2} \sin \alpha_{W_3} \sin \theta_{W_3}) {}^{W_0}\mathbf{z}_{W_3} - (d_{W_2} \cos \alpha_{W_3} \sin \theta_{W_3}) {}^{W_0}\mathbf{y}_{W_3}. \quad (\text{A-6})$$

We observe that the coefficients of the expansion in (A-6) are *not* constant. The wrist, therefore, must be such that $d_{W_2} = 0$ in order for condition (A-2) to be satisfied.

The fifth and sixth vector terms in (A-3),

$$r_{W_3} {}^{W_0}\mathbf{z}_{W_2} = (r_{W_3} \cos \alpha_{W_3}) {}^{W_0}\mathbf{z}_{W_3} + (r_{W_3} \sin \alpha_{W_3}) {}^{W_0}\mathbf{y}_{W_3} \quad \text{and} \quad (d_{W_3}) {}^{W_0}\mathbf{x}_{W_3}, \quad (\text{A-7})$$

are readily expressed in terms of the basis $({}^{W_0}\mathbf{x}_{W_3}, {}^{W_0}\mathbf{y}_{W_3}, {}^{W_0}\mathbf{z}_{W_3})$ and do not generate additional constraints.

In summary, the wrist geometry that will satisfy (A-1) must be such that

$$r_{W_1} = d_{W_1} = r_{W_2} = d_{W_2} = 0. \quad (\text{A-8})$$

The conditions in (A-8) correspond to the spherical wrist geometry; i.e., the three axes of the wrist intersect at a common point ($d_{W_1} = r_{W_2} = d_{W_2} = 0$) and the origin of the wrist reference frame W_0 coincides with the common point of intersection ($r_{W_1} = 0$). Indeed, for the spherical wrist geometry in (A-8) the vector function in (A-2) becomes:

$$\mathbf{g}_t[\mathbf{g}_r^{-1}({}^{W_0}\mathbf{R}_{W_3})] = (d_{W_3}) {}^{W_0}\mathbf{x}_{W_3} + (r_{W_3} \sin \alpha_{W_3}) {}^{W_0}\mathbf{y}_{W_3} + (r_{W_3} \cos \alpha_{W_3}) {}^{W_0}\mathbf{z}_{W_3} \quad (\text{A-9})$$

and the associative property for multiplication in (A-1) is satisfied.

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